Cartesianism

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Preface

This is about an exercise in analytic geometry, created for my students (and myself) after the universities in Turkey were closed against the coronavirus. The exercise may be useful in several ways:

- It allows for, and encourages, checking one's own work.
- It is resistant to cheating.
- It can give students a sense of the teacher's job, since it involves writing one's own exercise.
- It reveals the information and beauty that can be hidden within an equation.

In the presentation here, I attempt to remove every difficulty that is not inherent in the exercise. In particular, before the final, supplementary section, I work almost exclusively with numerical parameters, rather than with letters in place of them.

Anybody working the exercise for themself will use their own numbers, or even letters if the solver cares to find a general solution.

The numbers just mentioned are the coordinates of three points in the plane, the line through the first two not containing the origin. The points determine an ellipse and an hyperbola, the equation of either of which can be written out in the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

This equation will be analyzed to reveal the axes of the conic.

As far as I can tell, if such an exercise is performed in existing analytic geometry textbooks, it is done by change of coordinates; but not here.

The reader should know about equations for lines and circles in the Cartesian or coordinate plane; and about rationalization of denominators, whereby

$$\frac{1}{a - \sqrt{b}} = \frac{a + \sqrt{b}}{a^2 - b},$$

if indeed $a^2 \neq b$; but as far as I can tell, nothing else is required, beyond the mathematics that one will naturally have learned, along with those particular topics.

Contents

List of Figures

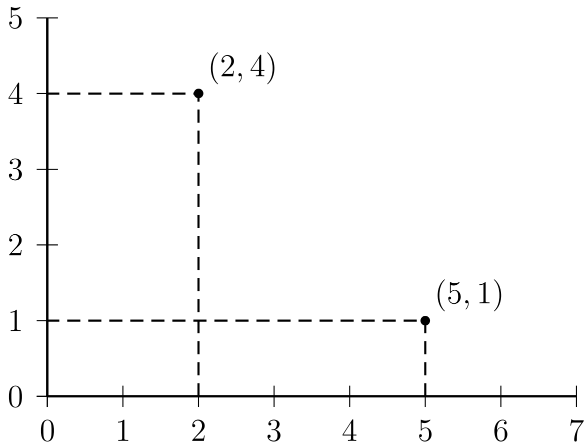

Figure 1: Two points

1 Points and parallelogram

We make a list of six integers, composing three ordered pairs. Let us work with

$$(5,1), \qquad (2,4), \qquad (4,1)$$

as an example. One should understand that these three ordered pairs correspond to three points in the so-called Cartesian plane. We shall not use the last point for a while. The first two points can be depicted as in Figure 1. Those points should be chosen so that the line through them does not pass through the origin, $(0,0)$. The coordinates of the points need not all be positive. Our work will be more interesting if none of the coordinates is 0.

Each of our first two points determines the line that passes through itself and the origin. These lines are defined by the equations

$$4x - 2y = 0, \qquad\qquad x - 5y = 0. \tag{1}$$

Any lines parallel to these will be defined by equations

$$4x - 2y = \alpha, \qquad\qquad x - 5y = \beta \tag{2}$$

respectively, for some α and β. Again, each of the lines defined in (1) passes through one of our two chosen points; the lines parallel to these, each passing through the *other* of our two points, are defined by

$$4x - 2y = 18, \qquad\qquad x - 5y = -18. \tag{3}$$

This is just because $4x - 2y$ takes the value 18 at $(5, 1)$, and $x - 5y$ takes the value -18 at $(2, 4)$. It is not accidental that these two values have the same absolute value. Our four lines now include the sides of a parallelogram, as in Figure 2.

2 Ellipse

We can factorize as $2(2x-y)$ the linear polynomial $4x-2y$ that appears in in (1), (2), and (3); and then we can simplify the corresponding equations; but it is probably better *not* to simplify at this point, because we are going to use the equations as they are to form

$$(4x - 2y)^2 + (x - 5y)^2 = 18^2. \tag{4}$$

This equation defines the kind of curve called an *ellipse,* but that is just a word. The equation is satisfied by the first two

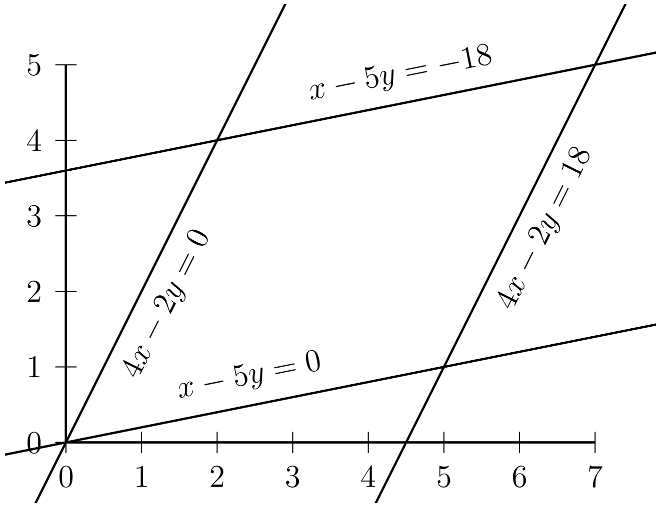

Figure 2: Parallelogram

of our chosen points; that is, those points lie on the ellipse. So does, for example, the point $(7/5, -13/5)$, which is $(1.4, -2.6)$, because it lies at the intersection of the lines defined in (2) above, provided

$$\alpha = 4 \cdot \frac{7}{5} - 2 \cdot \frac{-13}{5} = \frac{54}{5} = 10.8,$$
$$\beta = \frac{7}{5} - 5 \cdot \frac{-13}{5} = \frac{72}{5} = 14.4,$$

and here

$$\left(\frac{54}{5}\right)^2 + \left(\frac{72}{5}\right)^2 = \left(\frac{18}{5}\right)^2 (3^2 + 4^2) = 18^2.$$

Consequently, the point $(-7/5, 13/5)$ also lies on the ellipse. There is additional symmetry. The intersection of the lines

$$4x - 2y = \frac{54}{5}, \qquad x - 5y = \frac{-72}{5}$$

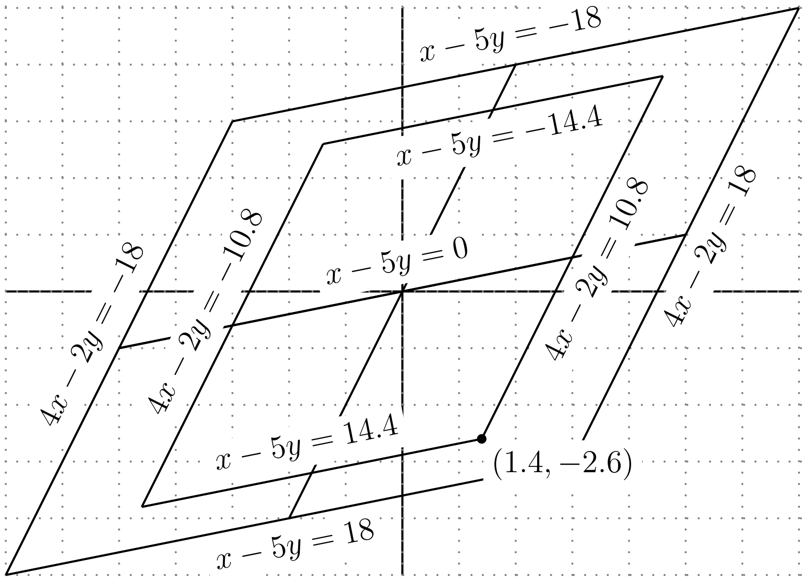

Figure 3: Nested parallelograms

also lies on the ellipse; and so does the intersection of

$$4x - 2y = \frac{-54}{5}, \qquad x - 5y = \frac{72}{5}.$$

For any α, the parallel lines defined by

$$4x - 2y = \alpha, \qquad 4x - 2y = -\alpha$$

lie at the same distance from the parallel line through the origin, defined by $4x - 2y = 0$. Likewise for lines parallel to $x - 5y = 0$. The specific lines that we have seen so far are depicted as in Figure 3, the one unlabelled line (besides the coordinate axes) being defined by $4x - 2y = 0$. Leaving out the equations, but drawing the ellipse that is defined by

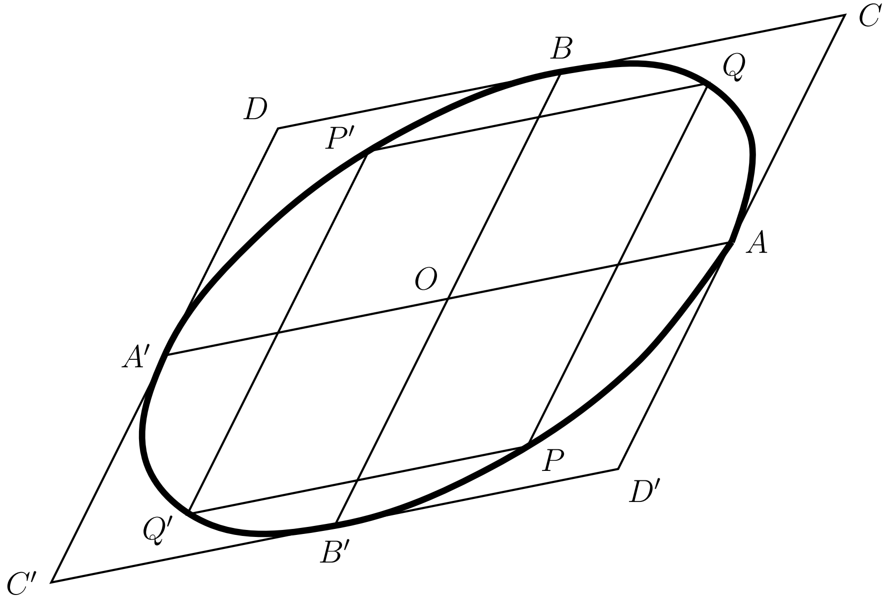

Figure 4: Ellipse

(4), and labelling some points with letters, we obtain Figure 4. In the letters of the diagram, the segments AA' and BB' are **diameters** of the ellipse, each **conjugate** to the other, because

- of the chords, such as PQ and $P'Q'$, that are drawn parallel to BB', the segment AA' bisects each of them;
- of the chords, such as PQ' and $P'Q$, that are drawn parallel to AA', the segment BB' bisects each of them.

The intersection O of the diameters is the **center** of the ellipse. The endpoints of the diameters are **vertices** of the ellipse. We may refer to endpoints of conjugate diameters as conjugate vertices. It will be useful to refer to the parallelogram

$CDC'D'$, whose sides contain the vertices and are parallel to the diameters, as a **bounding box** of the ellipse.

3 Expansion

If the ellipse has two conjugate diameters that are at right angles to one another, these are the **axes** of the ellipse. We shall prove that the axes exist by finding their equations. Our method is first to expand the squares in (4) and combine like terms, obtaining the equation

$$17x^2 - 26xy + 29y^2 = 18^2. \tag{5}$$

This is the same equation as (4), in the sense that the right-hand members are the same constant, and the lefthand members represent the same polynomial, albeit in different ways.

4 Completion of squares

We can rewrite the equation again by *completing squares.* Scaling the multiples of x^2 and xy in (4) for convenience, we compute

$$\begin{aligned} 17(17x^2 - 26xy) &= (17x)^2 - 2(17x)(13y) \\ &= (17x)^2 - 2(17x)(13y) + (13y)^2 - (13y)^2 \\ &= (17x - 13y)^2 - (13y)^2. \end{aligned}$$

Since $17 \cdot 29 - 13^2 = 493 - 169 = 324 = 18^2$, we compute now the identity

$$17(17x^2 - 26xy + 29y^2) = (17x - 13y)^2 + (18y)^2.$$

Therefore the ellipse defined by (4) and (5) is defined also by

$$(17x - 13y)^2 + (18y)^2 = 17 \cdot 18^2. \tag{6}$$

This is not the same equation as the others, but is equivalent, having only been scaled by 17. From (6) we can infer that the ellipse has conjugate diameters that are segments of the lines defined by

$$18y = 0, \qquad\qquad 17x - 13y = 0.$$

In particular, one of the diameters, defined by $18y = 0$ or more simply $y = 0$, is horizontal. For the endpoints of that diameter, we solve $y = 0$ simultaneously with (6), obtaining

$$\begin{aligned}(17x)^2 &= 17 \cdot 18^2,\\ 17x &= \pm 18\sqrt{17},\\ x &= \frac{\pm 18\sqrt{17}}{17}.\end{aligned}$$

Thus the ellipse has vertices

$$\left(\frac{\pm 18\sqrt{17}}{17}, 0\right).$$

Let us call them E and E'; one may use a calculator to find that these points are about $(\pm 4.37, 0)$. For the conjugate vertices, we solve $17x - 13y = 0$ simultaneously with (6), obtaining

$$\begin{aligned}(18y)^2 &= 17 \cdot 18^2,\\ y &= \pm\sqrt{17},\\ 17x = 13y &= \pm 13\sqrt{17},\\ x &= \frac{\pm 13\sqrt{17}}{17}.\end{aligned}$$

Thus the vertices conjugate to the ones that we already found are

$$\left(\frac{\pm 13\sqrt{17}}{17}, \pm\sqrt{17}\right).$$

Let us call them F and F'; they are about $(\pm 3.15, \pm 4.12)$. We confirm our computations by checking, as in Figure 5, that E

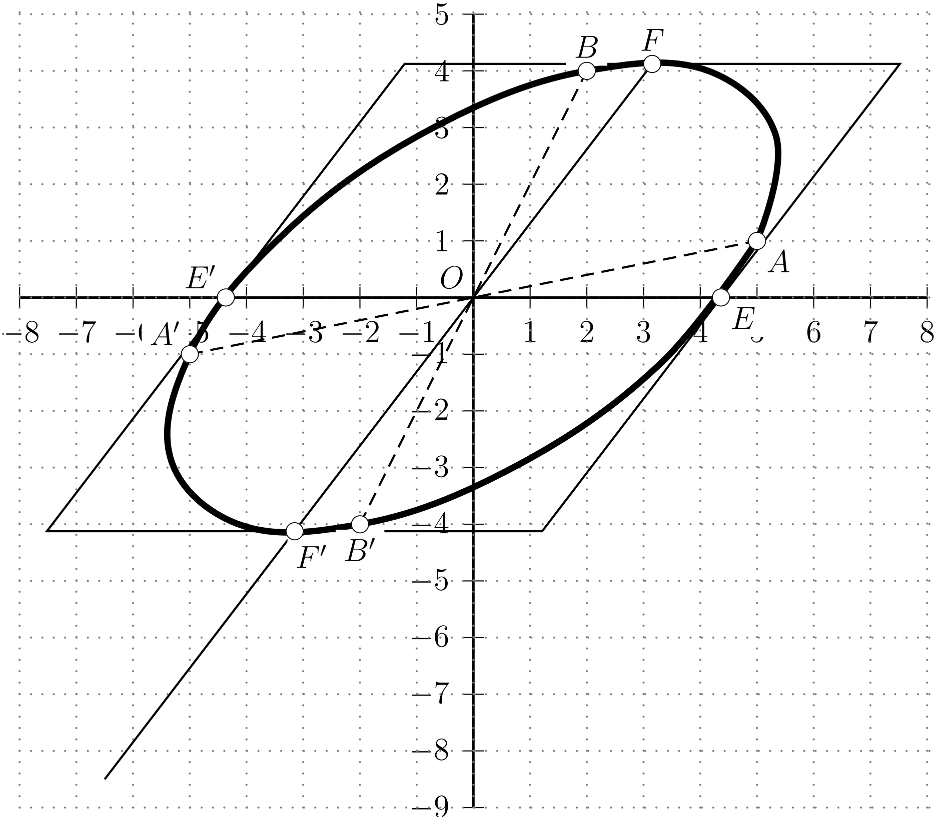

Figure 5: Ellipse with horizontal diameter

and F fit the graph that we already have, in the sense that the points do seem to lie on the graph, and what would be the corresponding bounding box, with sides defined respectively by the four equations

$$y = \pm\sqrt{17}, \qquad\qquad 17x - 13y = \pm 18 \cdot \sqrt{17},$$

does seem to be such. (We have extended FF' to pass, as it should, through $(-13/2, -17/2)$, which is $(-6.5, -8.5)$.)

5 Vertical equations

From equation (4), we can read off the equations in (1), which define lines of which conjugate diameters are segments; and we can also read off the coordinates of the corresponding vertices. Let us then refer to (4) as the **vertical equation** for those diameters. Then (6) is not a vertical equation, but it becomes vertical if we scale by 1/17 to obtain

$$\left(x\sqrt{17} - y\frac{13\sqrt{17}}{17}\right)^2 + \left(y\frac{18\sqrt{17}}{17}\right)^2 = 18^2.$$

Regardless of the diameters displayed, the constant term of any vertical equation for our ellipse will be 18^2: we shall not be able to prove this until the last section, but meanwhile, one can verify it in any particular case.

6 Slopes of axes

We are on our way to finding the axes of the ellipse. The circle that has diameter EE' will cut the ellipse also at some points G and G' as in Figure 6, and then, by the symmetry of the ellipse, the center of GG' will be that of EE', namely

O. Thus $EGE'G'$ is a rectangle. We shall show that the lines through O parallel to the sides of the rectangle are the axes of the ellipse. To this end, we note that the circle just mentioned is defined by

$$17x^2 + 17y^2 = 18^2. \tag{7}$$

Solving this simultaneously with (5), we obtain

$$\begin{aligned} 17x^2 - 26xy + 29y^2 &= 17x^2 + 17y^2, \\ 12y^2 - 26xy &= 0, \\ y(6y - 13x) &= 0. \end{aligned}$$

The equation of OG is thus $6y - 13x = 0$ or

$$6y = 13x. \tag{8}$$

We can compute G and G' by scaling (7) and substituting from (8) to obtain

$$(6 \cdot 18)^2 = 17(6x)^2 + 17(6y)^2 = 17(6x)^2 + 17(13x)^2.$$

Since

$$6^2 + 13^2 = 36 + 169 = 205, \tag{9}$$

we obtain

$$(6 \cdot 18)^2 = 17 \cdot 205x^2,$$

and then

$$x\sqrt{17} \cdot \sqrt{205} = \pm 6 \cdot 18, \qquad y\sqrt{17} \cdot \sqrt{205} = \pm 13 \cdot 18.$$

Taking its coordinates to be positive, we can write G as

$$\left(6 \cdot \frac{\sqrt{205}}{205} \cdot \frac{18\sqrt{17}}{17}, 13 \cdot \frac{\sqrt{205}}{205} \cdot \frac{18\sqrt{17}}{17}\right);$$

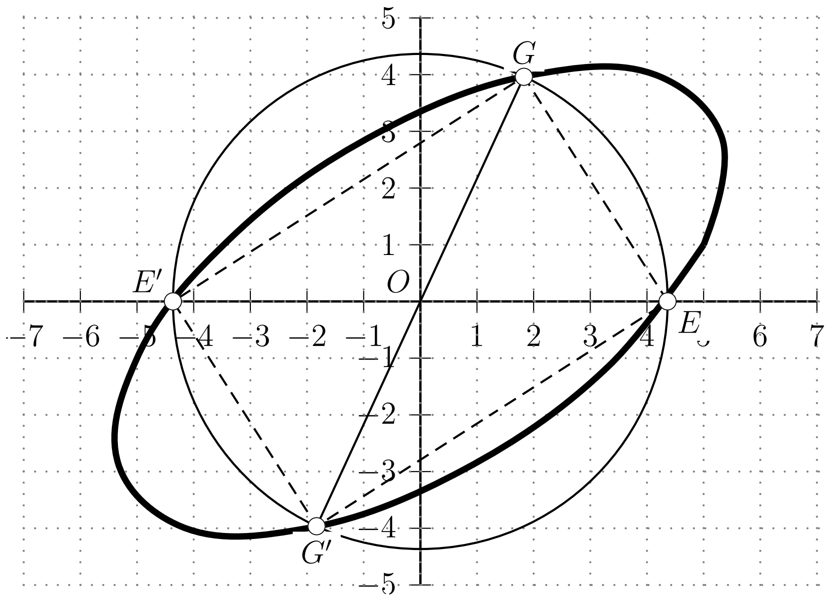

Figure 6: Ellipse and circle

this is about $(1.83, 3.96)$. Writing E and E' now as

$$\left(\pm\sqrt{205}\cdot\frac{\sqrt{205}}{205}\cdot\frac{18\sqrt{17}}{17},0\right),$$

We compute the slope of $E'G$ as either of

$$\frac{13}{6+\sqrt{205}}, \qquad \frac{6-\sqrt{205}}{-13},$$

and the slope of EG is either of

$$\frac{13}{6-\sqrt{205}}, \qquad \frac{6+\sqrt{205}}{-13}.$$

The product of the slopes is -1, as expected. Being parallel to $E'G$, the bisector of EOG is defined by either of

$$13x - (\sqrt{205} + 6)y = 0, \qquad (\sqrt{205} - 6)x - 13y = 0. \tag{10}$$

The bisector of EOG' is accordingly defined by either of

$$13y + (\sqrt{205} + 6)x = 0, \qquad (\sqrt{205} - 6)y + 13x = 0. \tag{11}$$

7 Vertical equation for axes

We shall show that the ellipse has axes that are segments of these bisectors. We could find the endpoints of a putative axis by solving the equation of its line simultaneously with one of our equations for the ellipse. We take the alternative approach of finding the vertical equation of the ellipse for the axes, in the sense defined above. If (α, β) and (γ, δ) are respectively endpoints of the axes that we expect to be defined by (10) and (11), then we expect the corresponding vertical equation to be

$$(\delta x - \gamma y)^2 + (\beta x - \alpha y)^2 = 18^2, \tag{12}$$

but also

$$-13\gamma = (\sqrt{205} - 6)\delta, \qquad 13\beta = (\sqrt{205} - 6)\alpha. \tag{13}$$

The point is to let the same choice from $\sqrt{205}-6$ and $\sqrt{205}+6$ appear in each of these two equations relating the coordinates of the vertices; we also have

$$-13\delta = (\sqrt{205} + 6)\gamma, \qquad 13\alpha = (\sqrt{205} + 6)\beta. \tag{14}$$

Using (13), we have

$$(\delta x - \gamma y)^2 + (\beta x - \alpha y)^2$$
$$= \frac{\delta^2\big(13x + (\sqrt{205} - 6)y\big)^2 + \alpha^2\big((\sqrt{205} - 6)x - 13y\big)^2}{13^2}.$$

Since we expect (5) and (12) to be the same as polynomial equations, we want to find α and δ so that

$$13^2(17x^2 - 26xy + 29y^2)$$
$$= \delta^2\big(13x + (\sqrt{205} - 6)\gamma\big)^2 + \alpha^2\big((\sqrt{205} - 6)x - 13y\big)^2.$$

By multiplying out and equating coefficients of x^2, xy, and y^2 respectively, we obtain the system

$$\left.\begin{aligned} 13^2\delta^2 + (\sqrt{205} - 6)^2\alpha^2 &= 13^2 \cdot 17, \\ 2 \cdot 13(\sqrt{205} - 6)(\delta^2 - \alpha^2) &= -13^2 \cdot 26, \\ (\sqrt{205} - 6)^2\delta^2 + 13^2\alpha^2 &= 13^2 \cdot 29 \end{aligned}\right\} \quad (15)$$

of three linear equations in the two unknowns δ^2 and α^2. If there turns out to be a solution, then we shall have achieved what we wanted. The middle equation simplifies to

$$(\sqrt{205} - 6)(\delta^2 - \alpha^2) = -13. \quad (16)$$

Now observe, using (9),

$$13^2 + (\sqrt{205} - 6)^2 = 2 \cdot 205 - 2 \cdot 6\sqrt{205} = 2(\sqrt{205} - 6)\sqrt{205},$$
$$13^2 - (\sqrt{205} - 6)^2 = -2 \cdot 6^2 + 2 \cdot 6\sqrt{205} = 12(\sqrt{205} - 6).$$

The sum and difference of the first and last equations of (15) are now

$$\left.\begin{aligned} 2(\sqrt{205} - 6)\sqrt{205}(\delta^2 + \alpha^2) &= 13^2 \cdot 46, \\ 12(\sqrt{205} - 6)(\delta^2 - \alpha^2) &= -13^2 \cdot 12. \end{aligned}\right\} \quad (17)$$

From (9),

$$13^2 = 205 - 6^2 = (\sqrt{205} + 6)(\sqrt{205} - 6),$$

and so the system (17) simplifies to

$$\left.\begin{aligned} \delta^2 + \alpha^2 &= \frac{\sqrt{205} + 6}{\sqrt{205}} \cdot 23, \\ \delta^2 - \alpha^2 &= \frac{\sqrt{205} + 6}{\sqrt{205}} \cdot (-\sqrt{205}). \end{aligned}\right\} \tag{18}$$

Note that the second equation is equivalent to (16), and therefore (15) has a solution, which is the solution of (18), namely

$$\delta^2 = \frac{(\sqrt{205} + 6)(23 - \sqrt{205})}{2\sqrt{205}} = \frac{17\sqrt{205} - 67}{2\sqrt{205}},$$
$$\alpha^2 = \frac{(\sqrt{205} + 6)(23 + \sqrt{205})}{2\sqrt{205}} = \frac{29\sqrt{205} + 343}{2\sqrt{205}}.$$

From (13) and (14),

$$\gamma^2 = \frac{(\sqrt{205} - 6)(23 - \sqrt{205})}{2\sqrt{205}} = \frac{29\sqrt{205} - 343}{2\sqrt{205}},$$
$$\beta^2 = \frac{(\sqrt{205} - 6)(23 + \sqrt{205})}{2\sqrt{205}} = \frac{17\sqrt{205} + 67}{2\sqrt{205}},$$

and so, because α and β have the same sign, while γ and δ are of opposite sign, (12) is

$$\left(x\sqrt{\frac{17\sqrt{205} - 67}{2\sqrt{205}}} + y\sqrt{\frac{29\sqrt{205} - 343}{2\sqrt{205}}}\right)^2 + \left(x\sqrt{\frac{17\sqrt{205} + 67}{2\sqrt{205}}} - y\sqrt{\frac{29\sqrt{205} + 343}{2\sqrt{205}}}\right)^2 = 18^2.$$

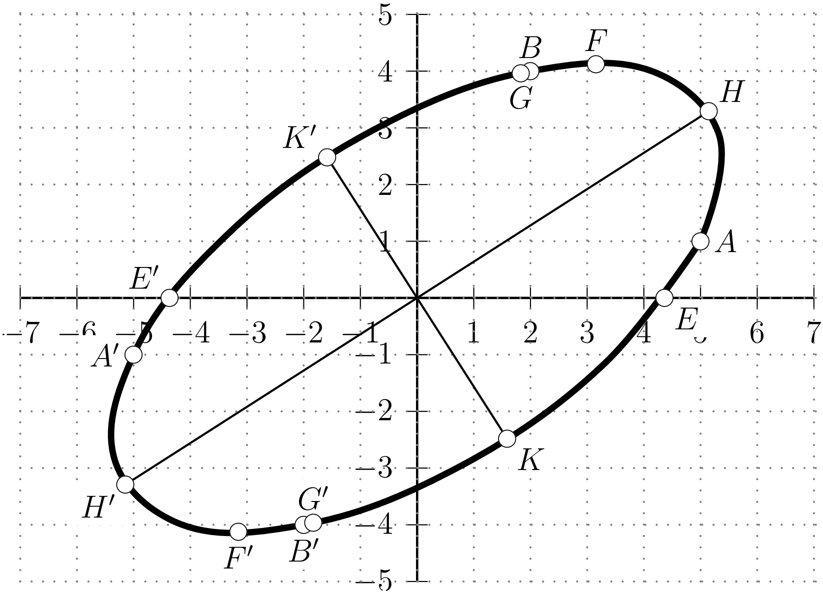

Figure 7: Ellipse and axes

Assuming α and γ are positive, we have

$$(\alpha, \beta) \approx (5.15, 3.29), \qquad (\gamma, \delta) \approx (1.59, -2.48).$$

Labelled as H and K, the points fit the graph as in Figure 7.

8 Translation

We haven't yet used the third point, namely $(4, 1)$, that we chose in the beginning. We can translate our ellipse by this, so that, instead of (4), its equation is

$$\big(4(x-4) - 2(y-1)\big)^2 + \big((x-4) - 5(y-1)\big)^2 = 18^2.$$

If we multiply this out, instead of (5) we get first

$$(4x - 2y - 14)^2 + (x - 5y + 1)^2 = 18^2,$$

and then

$$\begin{aligned}(4x - 2y)^2 - 28(4x - 2y) + 14^2 \\ + (x - 5y)^2 + 2(x - 5y) + 1 = 18^2,\end{aligned}$$

and finally

$$17x^2 - 26xy + 29y^2 - 110x + 46y = 127. \qquad (19)$$

We have thus created the exercise of analyzing the ellipse defined by (19). The first step of the solution is to complete the squares:

$$\begin{aligned}&17(17x^2 - 26xy - 110x) \\ &\qquad = (17x)^2 - 2(17x)(13y + 55) \\ &\qquad = (17x - 13y - 55)^2 - (13y + 55)^2,\end{aligned}$$

and then

$$\begin{aligned}&17(29y^2 + 46y) - (13y + 55)^2 \\ &\qquad = 493y^2 + 782y - 169y^2 - 1430y - 55^2 \\ &\qquad = 324y^2 - 648y - 3025 \\ &\qquad = 324(y^2 - 2y) - 3025 \\ &\qquad = 18^2(y - 1)^2 - 3349.\end{aligned}$$

Since $3349 = 17 \cdot 197$, and $197 + 127 = 324 = 18^2$, our equation (19) becomes

$$(17x - 13y - 55)^2 + 18(y - 1)^2 = 17 \cdot 18^2.$$

Since the lines defined by

$$17x - 13y - 55 = 0, \qquad\qquad y - 1 = 0$$

intersect at $(4, 1)$, our equation is also

$$\big(17(x-4) - 13(y-1)\big)^2 + \big(18(y-1)\big)^2 = 17 \cdot 18^2.$$

If we translate by $(-4, -1)$, then the curve is as defined by (6), and we can continue the analysis as before.

9 Vertical diameter

Working again with (5), we could first complete a square from the terms in y^2 and xy:

$$\begin{aligned} 29(29y^2 - 26xy) &= (29y)^2 - 2(29y)(13x) \\ &= (29y - 13x)^2 - (13x)^2. \end{aligned}$$

Since as before $29 \cdot 17 - 13^2 = 18^2$, we obtain for the ellipse the equations

$$(18x)^2 + (13x - 29y)^2 = 29 \cdot 18^2,$$
$$\left(x\frac{18\sqrt{29}}{29}\right)^2 + \left(x\frac{13\sqrt{29}}{29} - y\sqrt{29}\right)^2 = 18^2.$$

The latter is a vertical equation, and moreover one of the exhibited diameters is now vertical, rather than horizontal as before. Corresponding conjugate vertices are

$$\left(0, \frac{18\sqrt{29}}{29}\right), \qquad\qquad \left(\sqrt{29}, \frac{13\sqrt{29}}{29}\right),$$

which are about $(0, 3.34)$ and $(5.39, 2.41)$, as in Figure 8.

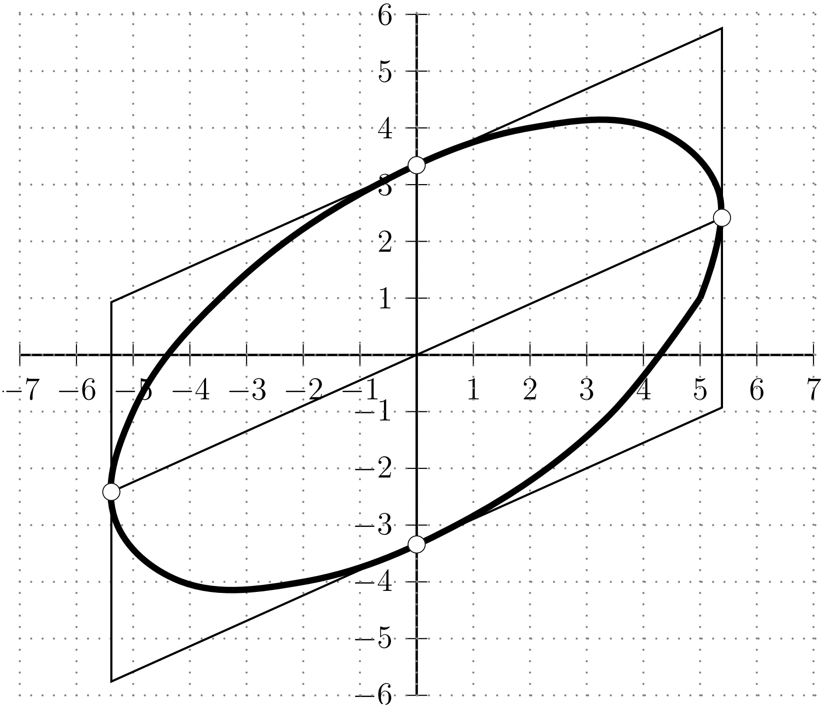

Figure 8: Ellipse with vertical diameter

10 Arbitrary diameters

Completing appropriate squares, we can establish the general claim that every chord of an ellipse that contains the center is a diameter, and there is another diameter that is conjugate to it. We have shown this for horizontal and vertical chords. A nonvertical line through the origin will be defined by an equation of the form $\alpha x - y = 0$. To find the conjugate diameter, we let $z = y - \alpha x$, so that $y = \alpha x + z$. We substitute in the equation and complete squares as before. Thus for example let $z = y - x$, so $y = x + z$. Then

$$\begin{aligned} 17x^2 - 26xy + 29y^2 &= 17x^2 - 26x(x+z) + 29(x+z)^2 \\ &= 20x^2 + 32xz + 29z^2, \end{aligned}$$

and then

$$\begin{gathered} 20x^2 + 32xz = 4(5x^2 + 8xz), \\ 5(5x^2 + 8xz) = (5x + 4z)^2 - 16z^2, \\ 5 \cdot 29 - 4 \cdot 16 = 145 - 64 = 81 = 9^2, \end{gathered}$$

and so (5) is equivalent to

$$\begin{gathered} 4(5x + 4z)^2 + (9z)^2 = 5 \cdot 18^2, \\ 4\big(5x + 4(y - x)\big)^2 + \big(9(y - x)\big)^2 = 5 \cdot 18^2, \\ 4(x + 4y)^2 + (9x - 9y)^2 = 5 \cdot 18^2, \\ \left(x\frac{2\sqrt{5}}{5} + y\frac{8\sqrt{5}}{5}\right)^2 + \left(x\frac{9\sqrt{5}}{5} - y\frac{9\sqrt{5}}{5}\right)^2 = 18^2. \end{gathered}$$

Thus conjugate axes are segments of the lines

$$x - y = 0, \qquad\qquad x + 4y = 0,$$

and corresponding conjugate vertices are

$$\left(\frac{9\sqrt{5}}{5}, \frac{9\sqrt{5}}{5}\right), \qquad\qquad \left(\frac{8\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\right),$$

which are about $(4.02, 4.02)$ and $(3.58, -0.89)$, as in Figure 9.

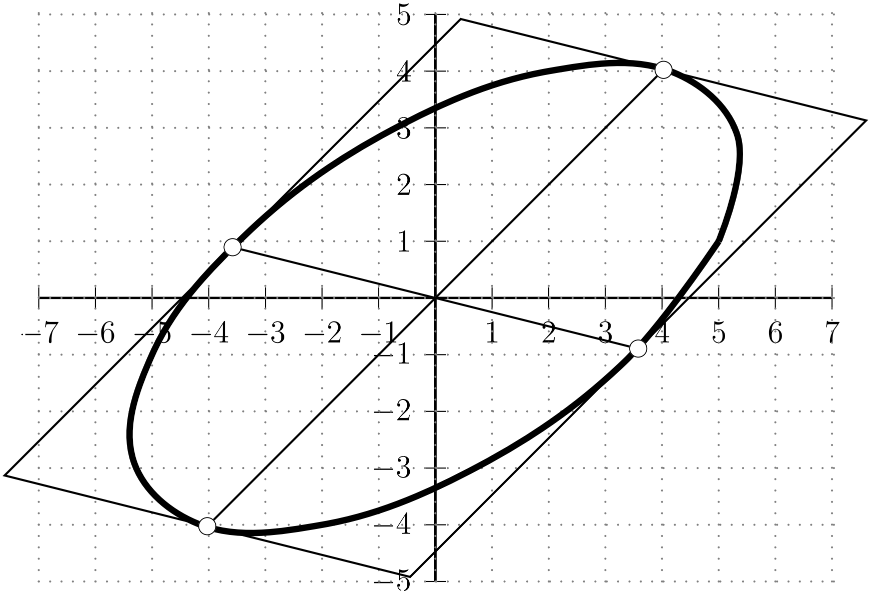

Figure 9: Ellipse with oblique diameter

11 Hyperbola

In place of (4), we can work with

$$(4x - 2y)^2 - (x - 5y)^2 = 18^2, \tag{20}$$

which defines the *hyperbola* shown in Figure 10. The same segments of of the lines defined by (1) are, by definition, conjugate diameters of the hyperbola, as they are of the ellipse. However, the endpoints of one of the diameters do not actually lie on the hyperbola. The argument of the previous section works here, establishing that every chord of an hyperbola that contains the center is a diameter and has a conjugate. Again though, a line through the center may not meet the hyperbola.

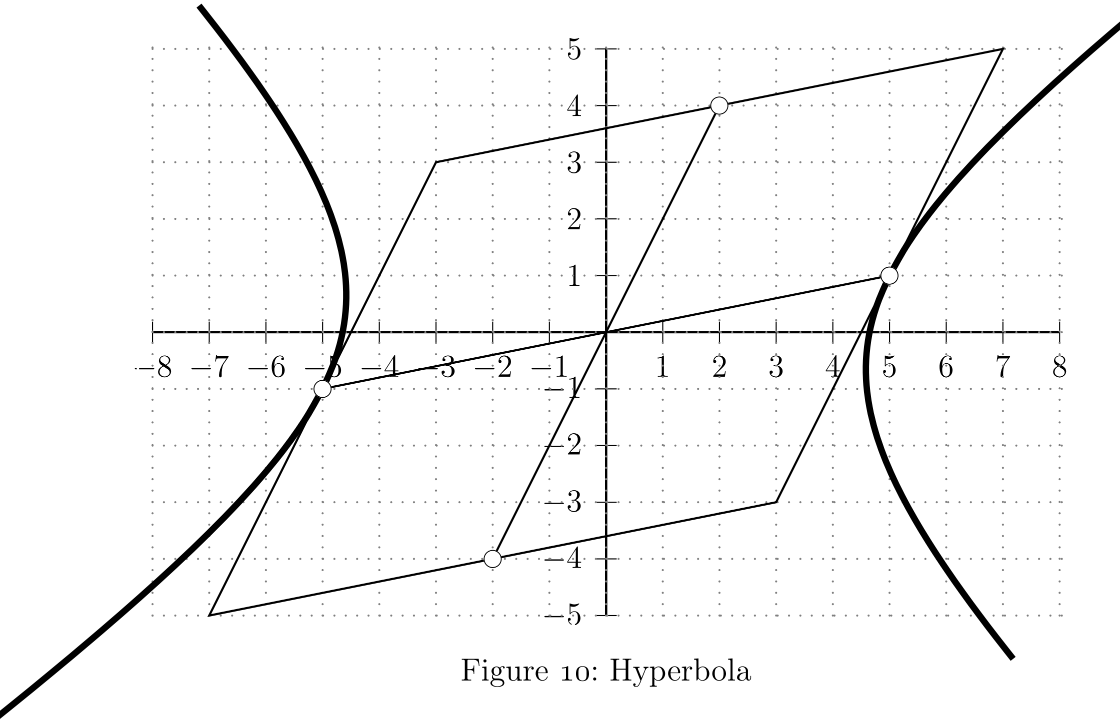

Figure 10: Hyperbola

It will still contain a diameter with a conjugate, unless it is an *asymptote,* meaning, in our example, one of the lines

$$4x - 2y = x - 5y, \qquad 4x - 2y = 5y - x.$$

We can still find axes, as we did for the ellipse. First, in place of (5), we obtain the defining equation

$$15x^2 - 6xy - 21y^2 = 18^2,$$

which reduces to

$$5x^2 - 2xy - 7y^2 = 3 \cdot 6^2, \tag{21}$$

though we shall have to remember that this is no longer a vertical equation. There is a horizontal diameter with endpoints

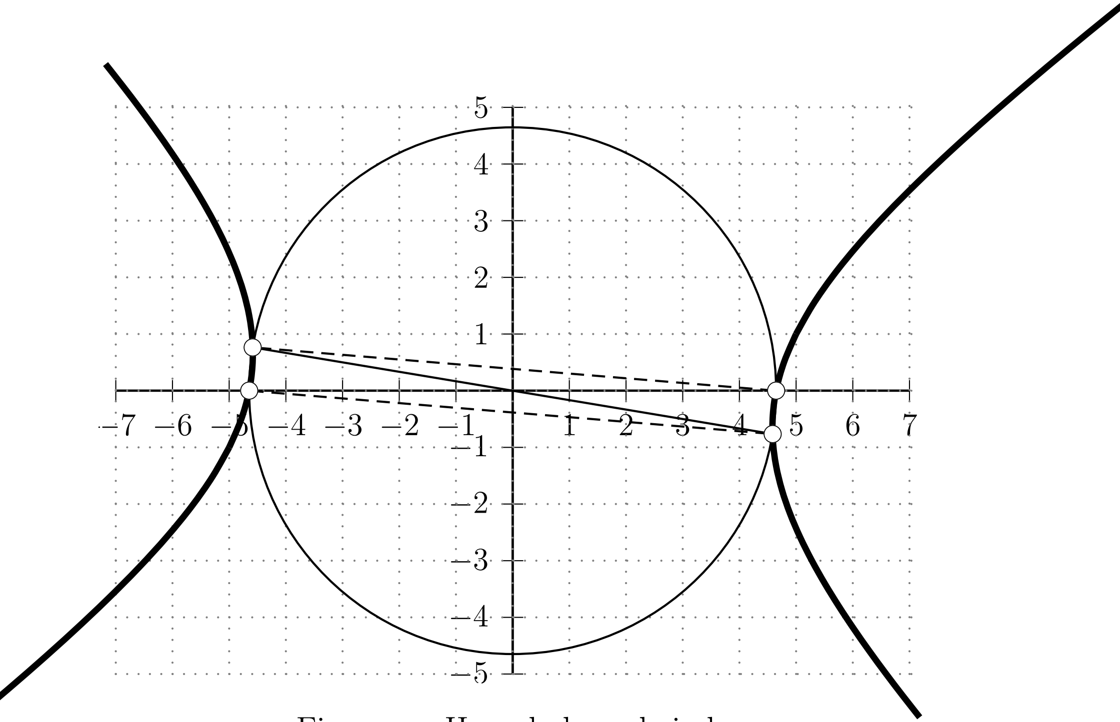

Figure 11: Hyperbola and circle

on the hyperbola, and the circle that shares this diameter is given by

$$5x^2 + 5y^2 = 3 \cdot 6^2.$$

The endpoints are

$$\left(\frac{\pm 6\sqrt{15}}{5}, 0 \right),$$

which are about $(\pm 4.64, 0)$, as in Figure 11. For the other common points, we solve

$$5x^2 - 2xy - 7y^2 = 5x^2 + 5y^2,$$
$$12y^2 + 2xy = 0,$$
$$y(6y + x) = 0.$$

The line given by $x+6y=0$ meets the hyperbola and circle where

$$3\cdot 6^2 = 5(6^2+1)y^2 = 5\cdot 37y^2.$$

Thus one of the meeting points is

$$\left(6\cdot\frac{\sqrt{37}}{37}\cdot\frac{6\sqrt{15}}{5},\frac{-\sqrt{37}}{37}\cdot\frac{6\sqrt{15}}{5}\right),$$

which are about $(\mp 4.58,\pm 0.76)$. The slopes of the axes must be

$$\frac{-1}{6+\sqrt{37}}, \qquad \frac{-1}{6-\sqrt{37}},$$

which are respectively

$$6-\sqrt{37}, \qquad 6+\sqrt{37}.$$

We thus expect for the hyperbola an equation

$$(\delta x-\gamma y)^2-(\beta x-\alpha y)^2=18^2, \tag{22}$$

where

$$\gamma=(\sqrt{37}-6)\delta, \qquad \beta=-(\sqrt{37}-6)\alpha,$$

so that

$$\begin{aligned}&(\delta x-\gamma y)^2-(\beta x-\alpha y)^2\\ &\qquad=\delta^2\big(x-(\sqrt{37}-6)y\big)^2-\alpha^2\big((\sqrt{37}-6)x+y\big)^2.\end{aligned}$$

Thus we want to solve

$$\left.\begin{aligned}\delta^2-(\sqrt{37}-6)\alpha^2&=15,\\ (\sqrt{37}-6)(\delta^2+\alpha^2)&=3,\\ (\sqrt{37}-6)^2\delta^2-\alpha^2&=-21.\end{aligned}\right\} \tag{23}$$

To this end, we compute

$$\begin{aligned} 1 + (\sqrt{37} - 6)^2 &= 2 \cdot 37 - 2 \cdot 6\sqrt{37} = 2(\sqrt{37} - 6)\sqrt{37}, \\ 1 - (\sqrt{37} - 6)^2 &= -2 \cdot 6^2 + 2 \cdot 6\sqrt{37} = 12(\sqrt{37} - 6), \end{aligned}$$

so that the first and last equations in (23) yield

$$\begin{aligned} \delta^2 - \alpha^2 &= \frac{\sqrt{37} + 6}{\sqrt{37}} \cdot (-3), \\ \delta^2 + \alpha^2 &= \frac{\sqrt{37} + 6}{\sqrt{37}} \cdot 3\sqrt{37}, \end{aligned}$$

and therefore

$$\begin{aligned} \delta^2 &= \frac{3(\sqrt{37} + 6)(\sqrt{37} - 1)}{2\sqrt{37}} = \frac{3(31 + 5\sqrt{37})}{2\sqrt{37}}, \\ \alpha^2 &= \frac{3(\sqrt{37} + 6)(\sqrt{37} + 1)}{2\sqrt{37}} = \frac{3(43 + 7\sqrt{37})}{2\sqrt{37}}, \\ \gamma^2 &= \frac{3(\sqrt{37} - 6)(\sqrt{37} - 1)}{2\sqrt{37}} = \frac{3(43 - 7\sqrt{37})}{2\sqrt{37}}, \\ \beta^2 &= \frac{3(\sqrt{37} - 6)(\sqrt{37} + 1)}{2\sqrt{37}} = \frac{3(31 - 5\sqrt{37})}{2\sqrt{37}}. \end{aligned}$$

We can now write (22) as

$$\begin{aligned} &\left(x\sqrt{\frac{3(31 + 5\sqrt{37})}{2\sqrt{37}}} - y\sqrt{\frac{3(43 - 7\sqrt{37})}{2\sqrt{37}}}\right)^2 \\ &\qquad - \left(x\sqrt{\frac{3(31 - 5\sqrt{37})}{2\sqrt{37}}} + y\sqrt{\frac{3(43 + 7\sqrt{37})}{2\sqrt{37}}}\right)^2 = 18^2. \end{aligned}$$

The axes are as in Figure 12; they are also axes of the conjugate hyperbola, given by

$$5x^2 - 2xy - 7y^2 = -3 \cdot 6^2.$$

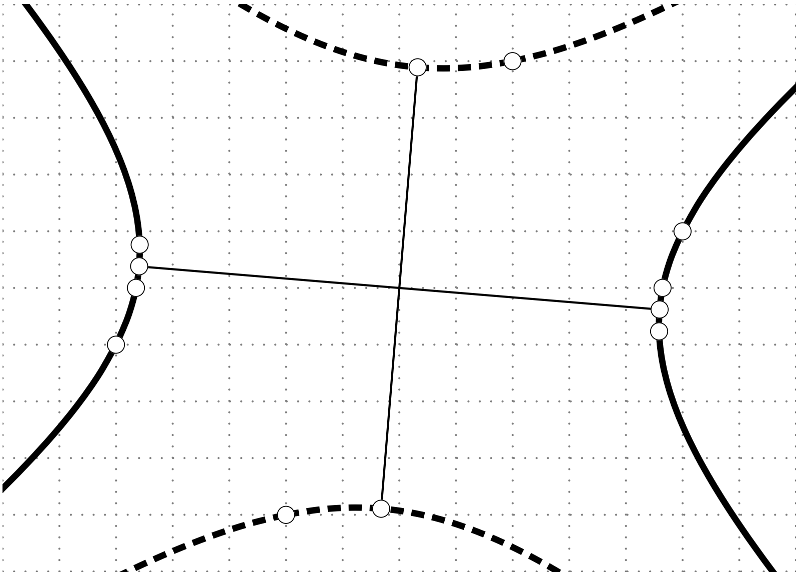

Figure 12: Conjugate hyperbolas and shared axes

12 Determinants

In (4), the number 18 is the **determinant** of the 2×2 matrix whose top row is $(5, 1)$ and whose bottom row is $(2, 4)$. In short,

$$18 = \det \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}.$$

In general, by definition,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Then, by computation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c + ta & d + tb \end{pmatrix}.$$

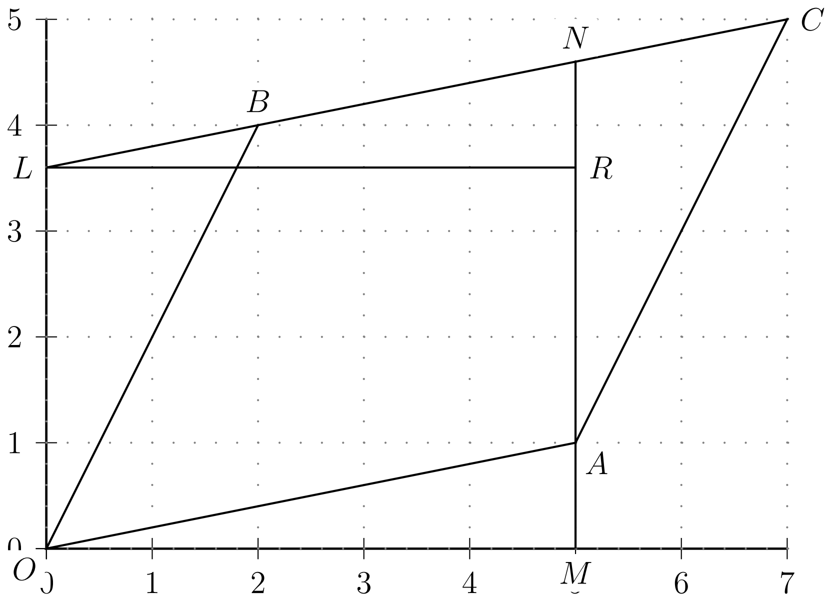

Figure 13: Determinant as area

If $a \neq 0$, then we may let $t = -c/a$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}.$$

Finally,

$$\begin{pmatrix} a & b \\ 0 & e \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}.$$

In our example, given points A and B as in Figure 13, we may slide B along the line parallel to OA until it reaches the y-axis as L, then slide A parallel to the y-axis until it reaches the x-axis as M. Three parallelograms are determined, all equal to one another:

$$OACB = OANL = OMRL.$$

Moreover, the last parallelogram is a rectangle. If the coordinates of A and B are (a, b) and (c, d), then the coordinates of L are $(0, d - bc/a)$, and the coordinates of M are $(a, 0)$. Thus the determinant $ad - bc$ is the area of the parallelogram $OACB$. The area is *signed*: interchanging the rows of the matrix changes the sign of the determinant. Using the abbreviation

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

we can write the vertical equation of the ellipse with center $(0, 0)$ and conjugate vertices (a_1, a_2) and (b_1, b_2) as

$$\begin{vmatrix} x & y \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ x & y \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2. \tag{24}$$

Let us now identify (a_1, a_2) with the **vector,** denoted by $\boldsymbol{a}$, represented by the directed segment from the origin to (a_1, a_2); and likewise, (b_1, b_2) with $\boldsymbol{b}$. We can add vectors and scale them. In (24), the first determinant is unchanged if (x, y) slides, parallel to $\boldsymbol{b}$, so as to meet $\boldsymbol{a}$; and the meeting point will be the endpoint of a vector $s \cdot \boldsymbol{a}$. likewise for the second determinant, if (x, y) slides parallel to $\boldsymbol{a}$ so as to meet $\boldsymbol{b}$: the meeting point will be the endpoint of a vector $t \cdot \boldsymbol{b}$. Thus, as in Figure 14, the ellipse consists of the points $s \cdot \boldsymbol{a} + t \cdot \boldsymbol{b}$, where

$$s^2 + t^2 = 1.$$

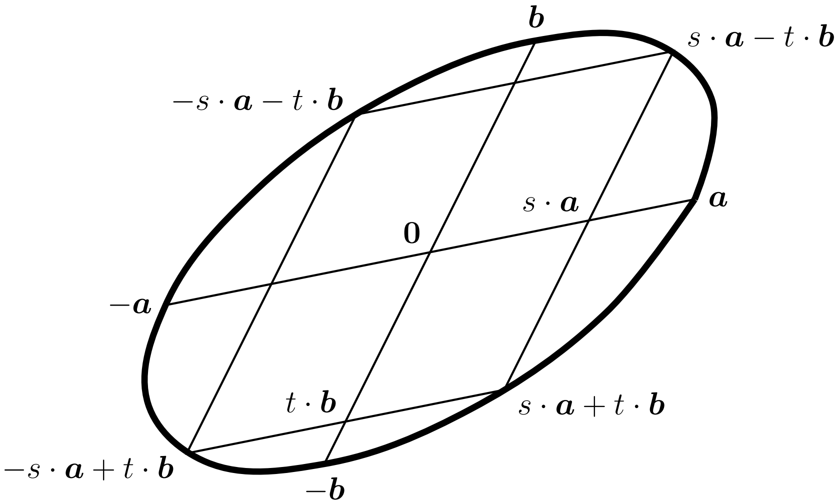

Figure 14: Ellipse