Cartesianism

David Pierce

August 2, 2020

Preface

This is about an exercise in analytic geometry, created for my students (and myself) after the universities in Turkey were closed against the coronavirus. The exercise may be useful in several ways:

- It allows for, and encourages, checking one's own work.
- It is resistant to cheating.
- It can give students a sense of the teacher's job, since it involves writing one's own exercise.
- It reveals the information and beauty that can be hidden within an equation.

In the presentation here, I attempt to remove every difficulty that is not inherent in the exercise. In particular, before the final, supplementary section, I work almost exclusively with numerical parameters, rather than with letters in place of them.

Anybody working the exercise for themself will use their own numbers, or even letters if the solver cares to find a general solution.

The numbers just mentioned are the coordinates of three points in the plane, the line through the first two not containing the origin. The points determine an ellipse and an hyperbola, the equation of either of which can be written out in the general form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

This equation will be analyzed to reveal the axes of the conic.

As far as I can tell, if such an exercise is performed in existing analytic geometry textbooks, it is done by change of coordinates; but not here.

The reader should know about equations for lines and circles in the Cartesian or coordinate plane; and about rationalization of denominators, whereby

$$\frac{1}{a - \sqrt{b}} = \frac{a + \sqrt{b}}{a^2 - b},$$

if indeed $a^2 \neq b$; but as far as I can tell, nothing else is required, beyond the mathematics that one will naturally have learned, along with those particular topics.

Contents

List of Figures

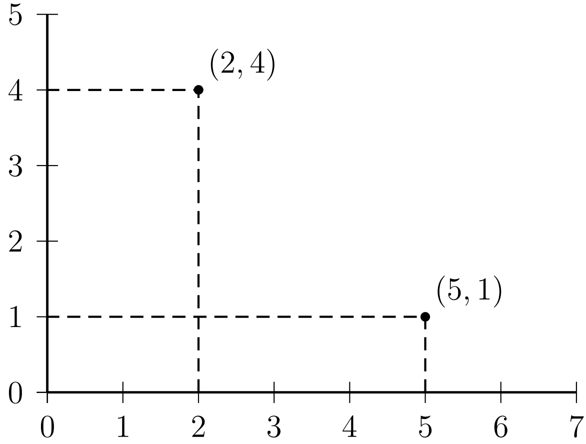

Figure 1: Two points

1 Points and parallelogram

We make a list of six integers, composing three ordered pairs. Let us work with

$$(5,1), \qquad (2,4), \qquad (4,1)$$

as an example. One should understand that these three ordered pairs correspond to three points in the so-called Cartesian plane. We shall not use the last point for a while. The first two points can be depicted as in Figure 1. Those points should be chosen so that the line through them does not pass through the origin, $(0,0)$. The coordinates of the points need not all be positive. Our work will be more interesting if none of the coordinates is 0.

Each of our first two points determines the line that passes through itself and the origin. These lines are defined by the equations

$$4x - 2y = 0, \qquad x - 5y = 0. \tag{1}$$

Any lines parallel to these will be defined by equations

$$4x - 2y = \alpha, \qquad x - 5y = \beta \tag{2}$$

respectively, for some α and β. Again, each of the lines defined in (1) passes through one of our two chosen points; the lines parallel to these, each passing through the *other* of our two points, are defined by

$$4x - 2y = 18, \qquad x - 5y = -18.$$

This is just because $4x - 2y$ takes the value 18 at $(5, 1)$, and $x - 5y$ takes the value -18 at $(2, 4)$. It is not accidental that these two values have the same absolute value. Our four lines now include the sides of a parallelogram, as in Figure 2.

2 Ellipse

We can simplify the equation $4x - 2y = 18$ to $2x - y = 9$, but it is probably better *not* to simplify at this point, because we are going to be interested in the equation

$$(4x - 2y)^2 + (x - 5y)^2 = 18^2. \tag{3}$$

This equation defines the kind of curve called an *ellipse,* but that is just a word. The equation is satisfied by the first two

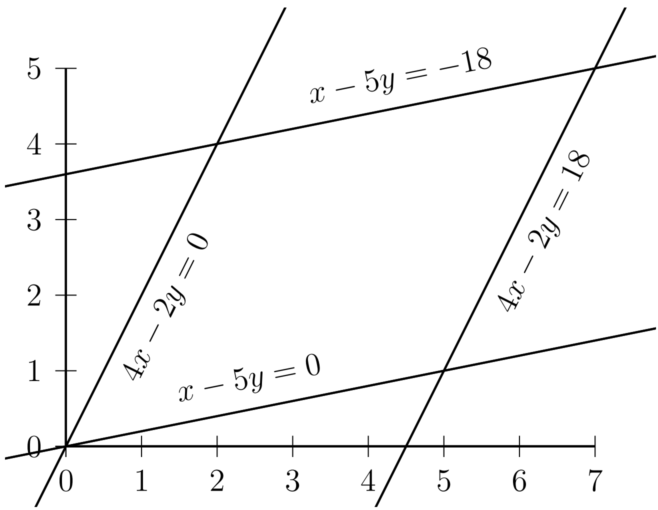

Figure 2: Parallelogram

of our chosen points; that is, those points lie on the ellipse. So does, for example, the point $(7/5, -13/5)$, which is $(1.4, -2.6)$. Consequently, the point $(-7/5, 13/5)$ also lies on the ellipse. There is additional symmetry. The point $(7/5, -13/5)$ is at the intersection of the lines defined in (2) above, provided

$$\alpha = 4 \cdot \frac{7}{5} - 2 \cdot \frac{-13}{5} = \frac{54}{5} = 10.8,$$
$$\beta = \frac{7}{5} - 5 \cdot \frac{-13}{5} = \frac{72}{5} = 14.4.$$

Then the intersection of the lines

$$4x - 2y = \frac{54}{5}, \qquad x - 5y = \frac{-72}{5}$$

also lies on the ellipse; and so does the intersection of

$$4x - 2y = \frac{-54}{5}, \qquad x - 5y = \frac{72}{5}.$$

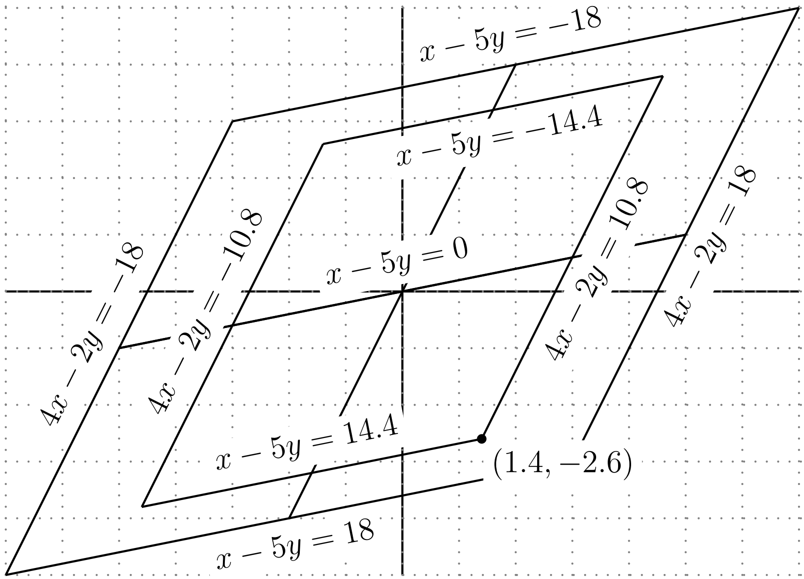

Figure 3: Nested parallelograms

For any α, the parallel lines defined by

$$4x - 2y = \alpha, \qquad\qquad 4x - 2y = -\alpha$$

lie at the same distance from the parallel line through the origin, defined by $4x - 2y = 0$. Likewise for lines parallel to $x - 5y = 0$. The specific lines that we have so far are depicted as in Figure 3, the one unlabelled line (besides the coordinate axes) being defined by $4x - 2y = 0$. Leaving out the equations, but drawing the ellipse that is defined by (3), and labelling some points with letters, we obtain Figure 4. In the letters of the diagram, the segments AA' and BB' are **diameters** of the ellipse, each **conjugate** to the other, because

- of the chords, such as PQ and $P'Q'$, that are drawn

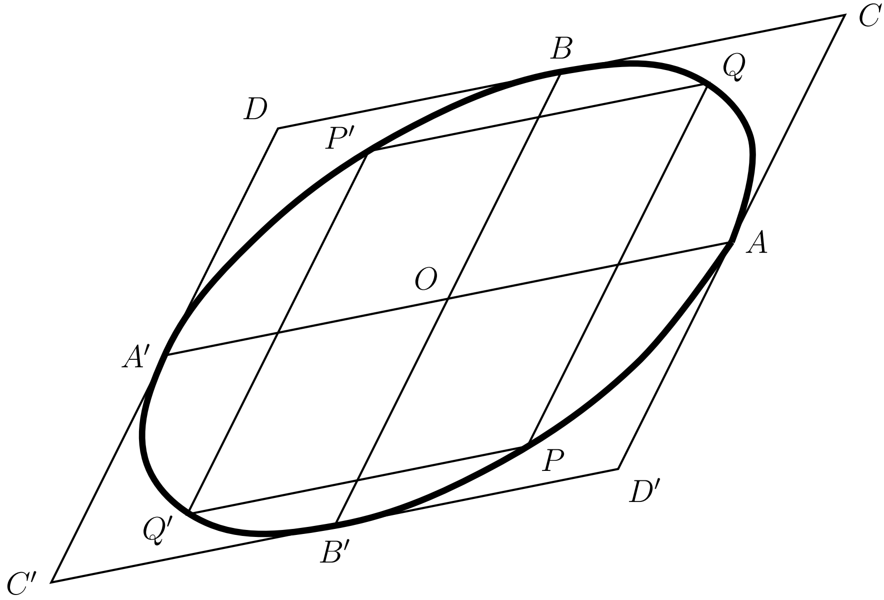

Figure 4: Ellipse

parallel to BB', the segment AA' bisects each of them;

- of the chords, such as PQ' and $P'Q$, that are drawn parallel to AA', the segment BB' bisects each of them.

The intersection O of the diameters is the **center** of the ellipse. The endpoints of the diameters are **vertices** of the ellipse. We may refer to endpoints of conjugate diameters as conjugate vertices. It will be useful to refer to the parallelogram $CDC'D'$, whose sides contain the vertices and are parallel to the diameters, as a **bounding box** of the ellipse.

3 Expansion

If the ellipse has two conjugate diameters that are at right angles to one another, these are the **axes** of the ellipse. We shall prove that the axes exist by finding their equations. Our method is first to expand the squares in (3) and combine like terms, obtaining the equation

$$17x^2 - 26xy + 29y^2 = 18^2. \tag{4}$$

This is the same equation as (3), in the sense that the righthand members are the same constant, and the lefthand members represent the same polynomial, albeit in different ways.

4 Completion of squares

We can rewrite the equation again by *completing squares*. Scaling the multiples of x^2 and xy in (3) for convenience, we compute

$$\begin{aligned} 17(17x^2 - 26xy) &= (17x)^2 - 2(17x)(13y) \\ &= (17x)^2 - 2(17x)(13y) + (13y)^2 - (13y)^2 \\ &= (17x - 13y)^2 - (13y)^2. \end{aligned}$$

Since $17 \cdot 29 - 13^2 = 493 - 169 = 324 = 18^2$, we compute now the identity

$$17(17x^2 - 26xy + 29y^2) = (17x - 13y)^2 + (18y)^2.$$

Therefore the ellipse defined by (3) and (4) is defined also by

$$(17x - 13y)^2 + (18y)^2 = 17 \cdot 18^2. \tag{5}$$

This is not the same equation as the others, but is equivalent, having only been scaled by 17. From (5) we can infer that the ellipse has conjugate diameters that are segments of the lines defined by

$$18y = 0, \qquad\qquad 17x - 13y = 0.$$

In particular, one of the diameters, defined by $18y = 0$ or more simply $y = 0$, is horizontal. For the endpoints of that diameter, we solve $y = 0$ simultaneously with (5), obtaining

$$\begin{aligned}(17x)^2 &= 17 \cdot 18^2,\\ 17x &= \pm 18\sqrt{17},\\ x &= \frac{\pm 18\sqrt{17}}{17} = \frac{\pm 18}{\sqrt{17}}.\end{aligned}$$

Thus the ellipse has vertices

$$\left(\frac{\pm 18\sqrt{17}}{17}, 0\right).$$

Let us call them E and E'; one may use a calculator to find that these points are about $(\pm 4.37, 0)$. For the conjugate vertices, we solve $17x - 13y = 0$ simultaneously with (5), obtaining

$$\begin{aligned}(18y)^2 &= 17 \cdot 18^2,\\ y &= \pm\sqrt{17},\\ 17x = 13y &= \pm 13\sqrt{17},\\ x &= \frac{\pm 13\sqrt{17}}{17}.\end{aligned}$$

Thus the vertices conjugate to the ones we found are

$$\left(\frac{\pm 13\sqrt{17}}{17}, \pm\sqrt{17}\right).$$

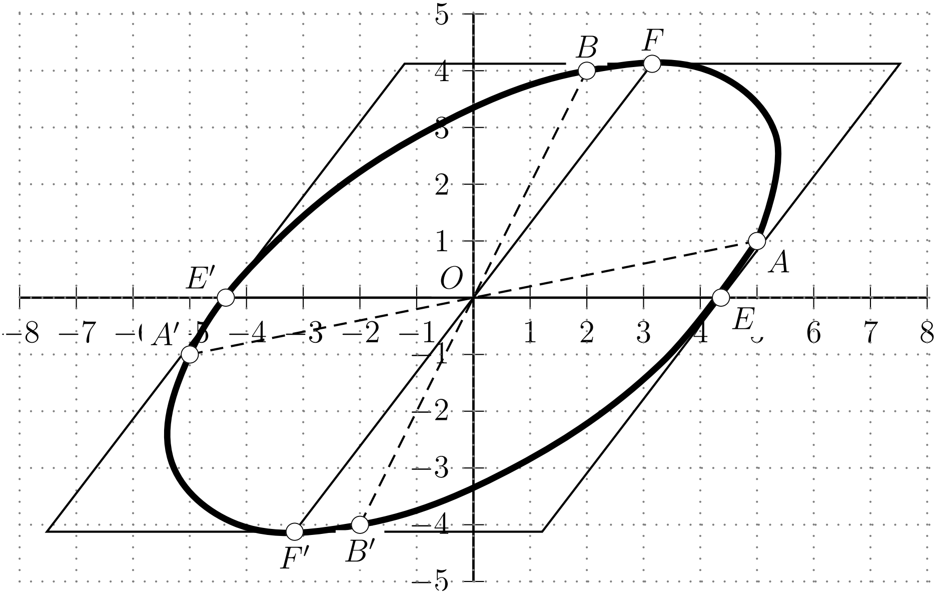

Figure 5: Ellipse with horizontal diameter

Let us call them F and F'; they are about $(\pm 3.15, \pm 4.12)$. We confirm our computations by checking, as in Figure 5, that E and F fit the graph that we already have, in the sense that the points do seem to lie on the graph, and what would be the corresponding bounding box, with sides defined by the four equations

$$y = \pm\sqrt{17}, \qquad 17x - 13y = \pm 18 \cdot \sqrt{17},$$

does seem to be such.

5 Vertical equations

From equation (3), we can read off the equations in (1), which define lines of which conjugate diameters are segments; and we can also read off the coordinates of the corresponding vertices. Let us then refer to (3) as the **vertical equation** for those diameters. Then (5) is not a vertical equation, but it becomes vertical if we scale by $1/17$ to obtain

$$\left(x\sqrt{17} - y\frac{13\sqrt{17}}{17}\right)^2 + \left(y\frac{18\sqrt{17}}{17}\right)^2 = 18^2.$$

Regardless of the diameters displayed, the constant term of any vertical equation for our ellipse will be 18^2: we shall not be able to prove this until the last section, but meanwhile, one can verify it in any particular case.

6 Slopes of axes

We are on our way to finding the axes of the ellipse. The circle that has diameter EE' will cut the ellipse also at some points G and G' as in Figure 6, and then, by the symmetry of the ellipse, the center of GG' will be that of EE', namely O. Thus $EGE'G'$ is a rectangle. We shall show that the lines through O parallel to the sides of the rectangle are the axes of the ellipse. To this end, we find the slope of OG by modifying (5) to obtain the equation

$$(17x)^2 + (17y)^2 = 17 \cdot 18^2, \tag{6}$$

which defines the circle just mentioned. Solving (6) together with (5) yields

$$(17x - 13y)^2 + (18y)^2 = (17x)^2 + (17y)^2,$$

$$
\begin{aligned}
-2\cdot 13\cdot 17xy + 13^2y^2 + 18^2y^2 &= 17^2y^2,\\
-442xy + (169+324-289)y^2 &= 0,\\
-442xy + 204y^2 &= 0,\\
y(204y - 442x) &= 0.
\end{aligned}
$$

The line defined by $y = 0$ is OE; so OG must be the line defined by $204y - 442x = 0$. Here we do want to simplify. One way to do this is the Euclidean Algorithm:

$$
\begin{aligned}
442 &= 204\cdot 2 + 34,\\
204 &= 34\cdot 6,
\end{aligned}
$$

and so the greatest common measure of 442 and 204 is 34. Since from our computations

$$442 = 34(6\cdot 2 + 1) = 34\cdot 13,$$

the equation of OG is $6y - 13x = 0$ or

$$6y = 13x. \tag{7}$$

We can compute G and G' by scaling (6) and substituting from (7) to obtain

$$
\begin{aligned}
(17\cdot 6x)^2 + (17\cdot 6y)^2 &= 17\cdot 6^2\cdot 18^2,\\
(17\cdot 6x)^2 + (17\cdot 13x)^2 &= 17\cdot 6^2\cdot 18^2,\\
17(6^2+13^2)x^2 &= 6^2\cdot 18^2,\\
x\sqrt{17}\cdot\sqrt{205} &= \pm 6\cdot 18,\\
y\sqrt{17}\cdot\sqrt{205} &= \pm 13\cdot 18.
\end{aligned}
$$

Taking the coordinates of G to be positive, we have G in either of the forms

$$\left(\frac{6\cdot 18}{\sqrt{17}\cdot\sqrt{205}}, \frac{13\cdot 18}{\sqrt{17}\cdot\sqrt{205}}\right),$$

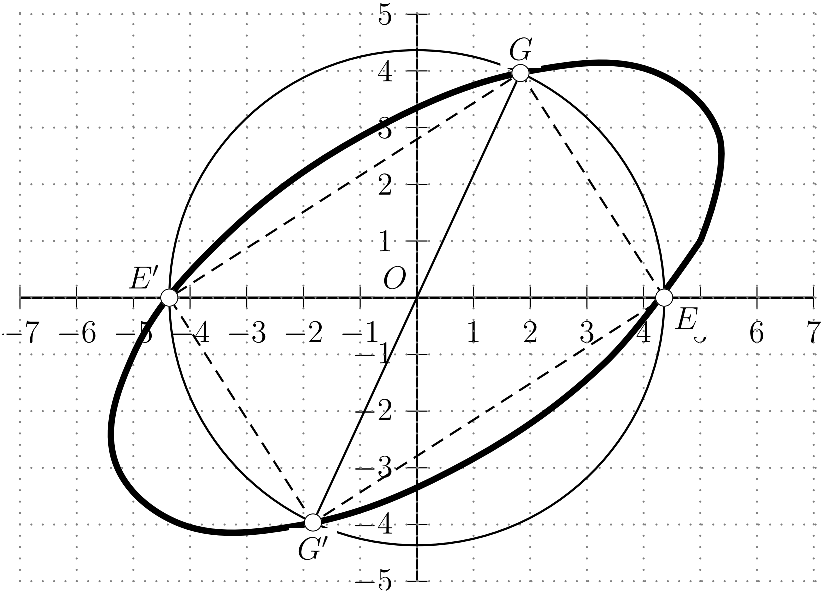

Figure 6: Ellipse and circle

$$\left(\frac{6 \cdot 18\sqrt{17} \cdot \sqrt{205}}{17 \cdot 205}, \frac{13 \cdot 18\sqrt{17} \cdot \sqrt{205}}{17 \cdot 205}\right);$$

this is about $(1.83, 3.96)$. We compute now the slope of $E'G$:

$$\frac{\dfrac{13 \cdot 18}{\sqrt{17} \cdot \sqrt{205}}}{\dfrac{18}{\sqrt{17}} + \dfrac{6 \cdot 18}{\sqrt{17} \cdot \sqrt{205}}} = \frac{13}{\sqrt{205} + 6}$$

$$= \frac{13(\sqrt{205} - 6)}{205 - 36} = \frac{\sqrt{205} - 6}{13}.$$

Each of the shorter two expressions for the slope will be useful. Because EGE' is a right angle, the slope of EG is either of

$$-\frac{\sqrt{205}+6}{13}, \qquad -\frac{13}{\sqrt{205}-6}.$$

The bisector of EOG is defined by either of

$$13y = (-6+\sqrt{205})x, \qquad (6+\sqrt{205})y = 13x,$$

and hence by either of

$$(\sqrt{205}-6)x - 13y = 0, \qquad 13x - (\sqrt{205}+6)y = 0. \tag{8}$$

The bisector of EOG' is accordingly defined by either of

$$13x + (\sqrt{205}-6)y = 0, \qquad (\sqrt{205}+6)x + 13y = 0. \tag{9}$$

7 Vertical equation for axes

We shall show that the ellipse has axes that are segments of these bisectors. We could find the endpoints of a putative axis by solving the equation of its line simultaneously with one of our equations for the ellipse. We take the alternative approach of finding the vertical equation of the ellipse for the axes, in the sense defined above. This equation will take the form

$$\alpha\bigl(13x + (\sqrt{205}-6)y\bigr)^2 + \beta\bigl((\sqrt{205}-6)x - 13y\bigr)^2 = 18^2 \tag{10}$$

for some α and β, where we have used the first equations in (9) and (8). We could have incorporated the other equation in either case, but the symmetry of (10) will be useful. We want to show that this equation and (4) define the same ellipse. Since the right members of the equations are the same, the left members should be the same polynomial. In particular, by multiplying out and equating coefficients of x^2, xy, and y^2 respectively, we obtain the system

$$\left.\begin{aligned} 13^2\alpha + (\sqrt{205} - 6)^2\beta = 17, \\ 2 \cdot 13(\sqrt{205} - 6)(\alpha - \beta) = -26, \\ (\sqrt{205} - 6)^2\alpha + 13^2\beta = 29 \end{aligned}\right\} \tag{11}$$

of three linear equations in the two unknowns α and β. If there turns out to be a solution, then we shall have achieved what we wanted. The middle equation simplifies to

$$(\sqrt{205} - 6)(\alpha - \beta) = -1. \tag{12}$$

Now let us define

$$\begin{aligned} \gamma &= 13^2 + (\sqrt{205} - 6)^2 \\ &= 13^2 + \left(\sqrt{6^2 + 13^2} - 6\right)^2 \\ &= 2(6^2 + 13^2) - 2 \cdot 6\sqrt{6^2 + 13^2} \\ &= 2\left(\sqrt{6^2 + 13^2} - 6\right)\sqrt{6^2 + 13^2} \\ &= 2(\sqrt{205} - 6)\sqrt{205}. \end{aligned}$$

Then (12) simplifies again to

$$\gamma(\alpha - \beta) = -2\sqrt{205}.$$

The system (11) has become

$$\begin{aligned} 13^2\alpha + (\gamma - 13^2)\beta &= 17, \\ \gamma\alpha - \gamma\beta &= -2\sqrt{205}, \\ (\gamma - 13^2)\alpha + 13^2\beta &= 29. \end{aligned}$$

Adding the first equation to the last gives the equivalent system

$$\begin{aligned} 13^2\alpha + (\gamma - 13^2)\beta &= 17,\\ \gamma\alpha - \qquad\qquad \gamma\beta &= -2\sqrt{205},\\ \gamma\alpha + \qquad\qquad \gamma\beta &= 46, \end{aligned}$$

which in turn is equivalent to

$$\left.\begin{aligned} 13^2\alpha + (\gamma - 13^2)\beta &= 17,\\ \gamma\alpha \qquad\qquad\qquad &= 23 - \sqrt{205},\\ \gamma\beta &= 23 + \sqrt{205}. \end{aligned}\right\} \qquad (13)$$

This has the obvious solution, since

$$13^2\alpha + (\gamma - 13^2)\beta = 13^2(\alpha - \beta) + \gamma\beta,$$

and if the last two equations of (13) are true, then

$$13^2(\alpha - \beta) = \frac{13^2}{\gamma} \cdot \gamma(\alpha - \beta) = \frac{-13^2}{\sqrt{205} - 6} = -(\sqrt{205} + 6),$$

so that the first equation of (13) is true. Now (10) is

$$\begin{aligned} &\frac{23 - \sqrt{205}}{\gamma}\big(13x + (\sqrt{205} - 6)y\big)^2\\ &\qquad + \frac{23 + \sqrt{205}}{\gamma}\big((\sqrt{205} - 6)x - 13y\big)^2 = 18^2. \end{aligned} \qquad (14)$$

Since

$$\gamma(\sqrt{205} + 6) = 2 \cdot 13^2\sqrt{205},$$

and

$$\begin{aligned} (23 - \sqrt{205})(\sqrt{205} + 6) &= (23 - 6)\sqrt{205} + 138 - 205\\ &= 17\sqrt{205} - 67, \end{aligned}$$

while

$$\begin{aligned}(23+\sqrt{205})(\sqrt{205}+6) &= (23+6)\sqrt{205}+138+205\\ &= 29\sqrt{205}+343,\end{aligned}$$

equation (14) for the ellipse becomes

$$\frac{17\sqrt{205}-67}{2\sqrt{205}}\left(x+\frac{\sqrt{205}-6}{13}y\right)^2 + \frac{29\sqrt{205}+343}{2\sqrt{205}}\left(\frac{\sqrt{205}-6}{13}x-y\right)^2 = 18^2. \quad (15)$$

This is a vertical equation for the axes of the ellipse.

8 Axes themselves

From (15) we can read off endpoints of the axes:

$$\left(\sqrt{\frac{29\sqrt{205}+343}{2\sqrt{205}}}, \frac{\sqrt{205}-6}{13}\sqrt{\frac{29\sqrt{205}+343}{2\sqrt{205}}}\right),$$
$$\left(\frac{\sqrt{205}-6}{13}\sqrt{\frac{17\sqrt{205}-67}{2\sqrt{205}}}, -\sqrt{\frac{17\sqrt{205}-67}{2\sqrt{205}}}\right).$$

Let us call these H and K respectively; their negatives, H' and K'. Their quadrants are thus:

K'	H
H'	K

We should like to simplify the second coordinate of H and the first coordinate of K. One can do this by brute force; we use symmetry. We make two observations.

1. In the equations (8), if we replace $\sqrt{205}$ with $-\sqrt{205}$, we obtain the equations (9).

2. In finding α and β such that (10), all we use about $\sqrt{205}$ is that its square is 205.

Therefore, in (15), if we replace $\sqrt{205}$ with $-\sqrt{205}$, then we shall still have the vertical equation for our ellipse, only now we read off conjugate vertices

$$\left(\sqrt{\frac{29\sqrt{205}-343}{2\sqrt{205}}}, -\frac{\sqrt{205}+6}{13}\sqrt{\frac{29\sqrt{205}+343}{2\sqrt{205}}}\right),$$
$$\left(-\frac{\sqrt{205}+6}{13}\sqrt{\frac{17\sqrt{205}+67}{2\sqrt{205}}}, -\sqrt{\frac{17\sqrt{205}+67}{2\sqrt{205}}}\right).$$

The first of these must be K; the second, H'; because of the quadrants they are in. Using the simpler expression for a coordinate in each case, we can now write H and K respectively as

$$\left(\sqrt{\frac{29\sqrt{205}+343}{2\sqrt{205}}}, \sqrt{\frac{17\sqrt{205}+67}{2\sqrt{205}}}\right),$$
$$\left(\sqrt{\frac{29\sqrt{205}-343}{2\sqrt{205}}}, -\sqrt{\frac{17\sqrt{205}-67}{2\sqrt{205}}}\right).$$

These are approximately

$$(5.15, 3.29), \qquad (1.59, -2.48),$$

and, labelled as H and K, they fit the graph as in Figure 7. We can also now rewrite (15) as

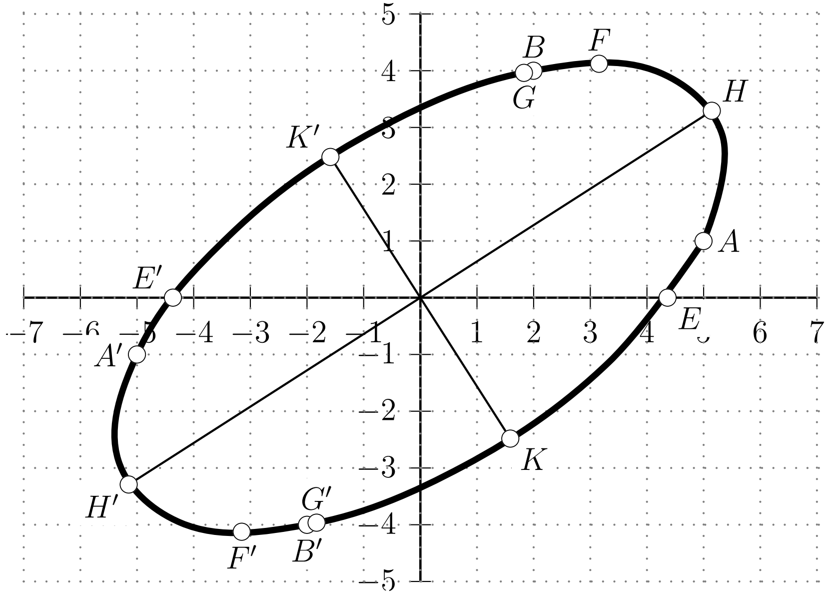

Figure 7: Ellipse and axes

$$\left(x\sqrt{\frac{17\sqrt{205}-67}{2\sqrt{205}}}+y\sqrt{\frac{29\sqrt{205}-343}{2\sqrt{205}}}\right)^2 + \left(x\sqrt{\frac{17\sqrt{205}+67}{2\sqrt{205}}}-y\sqrt{\frac{29\sqrt{205}+343}{2\sqrt{205}}}\right)^2 = 18^2.$$

9 Translation

We haven't yet used the third point, namely $(4, 1)$, that we chose in the beginning. We can translate our ellipse by this, so that, instead of (3), its equation is

$$\big(4(x-4)-2(y-1)\big)^2 + \big((x-4)-5(y-1)\big)^2 = 18^2.$$

If we multiply this out, instead of (4) we get first

$$(4x - 2y - 14)^2 + (x - 5y + 1)^2 = 18^2,$$

and then

$$(4x - 2y)^2 - 28(4x - 2y) + 14^2 \\ + (x - 5y)^2 + 2(x - 5y) + 1 = 18^2,$$

and finally

$$17x^2 - 26xy + 29y^2 - 110x + 46y = 127. \qquad (16)$$

We have thus created the exercise of analyzing the ellipse defined by (16). The first step of the solution is to complete the squares:

$$\begin{aligned} 17(17x^2 - 26xy - 110x) \\ &= (17x)^2 - 2(17x)(13y + 55) \\ &= (17x - 13y - 55)^2 - (13y + 55)^2, \end{aligned}$$

and then

$$\begin{aligned} 17(29y^2 + 46y) - (13y + 55)^2 \\ &= 493y^2 + 782y - 169y^2 - 1430y - 55^2 \\ &= 324y^2 - 648y - 3025 \\ &= 324(y^2 - 2y) - 3025 \\ &= 18^2(y - 1)^2 - 3349. \end{aligned}$$

Since $3349 = 17 \cdot 197$, and $197 + 127 = 324 = 18^2$, our equation (16) becomes

$$(17x - 13y - 55)^2 + 18(y - 1)^2 = 17 \cdot 18^2.$$

Since the lines defined by

$$17x - 13y - 55 = 0, \qquad\qquad y - 1 = 0$$

intersect at $(4, 1)$, our equation is also

$$\bigl(17(x-4) - 13(y-1)\bigr)^2 + \bigl(18(y-1)\bigr)^2 = 17 \cdot 18^2.$$

If we translate by $(-4, -1)$, then the curve is as defined by (5), and we can continue the analysis as before.

10 Vertical diameter

Working again with (4), we could first complete a square from the terms in y^2 and xy:

$$\begin{aligned} 29(29y^2 - 26xy) &= (29y)^2 - 2(29y)(13x) \\ &= (29y - 13x)^2 - (13x)^2. \end{aligned}$$

Since as before $29 \cdot 17 - 13^2 = 18^2$, we obtain for the ellipse the equations

$$(18x)^2 + (13x - 29y)^2 = 29 \cdot 18^2,$$
$$\left(x\frac{18\sqrt{29}}{29}\right)^2 + \left(x\frac{13\sqrt{29}}{29} - y\sqrt{29}\right)^2 = 18^2.$$

The latter is a vertical equation, and moreover one of the exhibited diameters is now vertical, rather than horizontal as before. Corresponding conjugate vertices are

$$\left(0, \frac{18\sqrt{29}}{29}\right), \qquad\qquad \left(\sqrt{29}, \frac{13\sqrt{29}}{29}\right),$$

which are about $(0, 3.34)$ and $(5.39, 2.41)$, as in Figure 8.

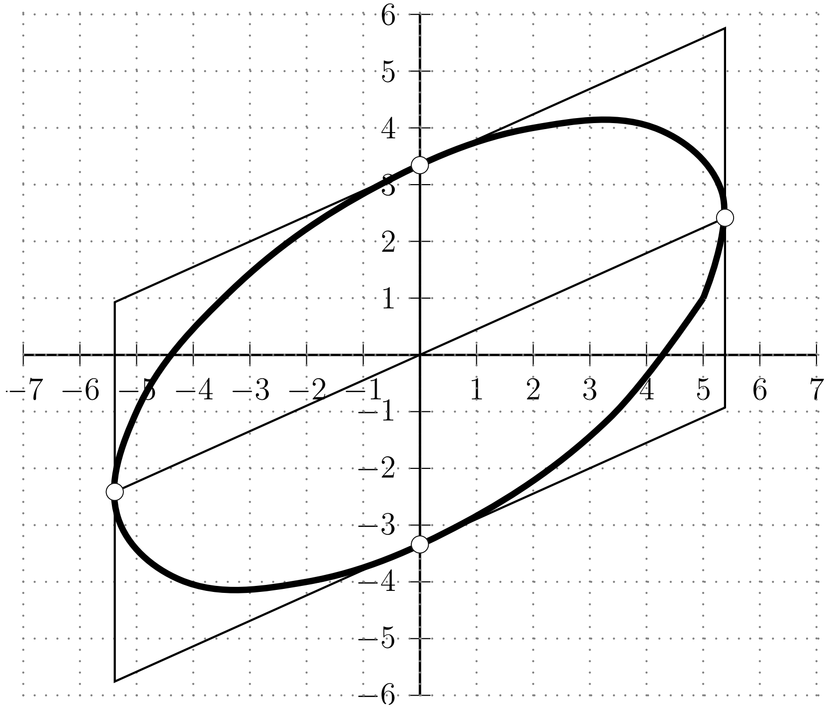

Figure 8: Ellipse with vertical diameter

11 Arbitrary diameters

Completing appropriate squares, we can establish the general claim that every chord of an ellipse that contains the center is a diameter, and there is another diameter that is conjugate to it. We have shown this for horizontal and vertical chords. A nonvertical line through the origin will be defined by an equation of the form $\alpha x - y = 0$. To find the conjugate diameter, we let $z = y - \alpha x$, so that $y = \alpha x + z$. We substitute in the equation and complete squares as before. Thus for example let $z = y - x$, so $y = x + z$. Then

$$\begin{aligned} 17x^2 - 26xy + 29y^2 &= 17x^2 - 26x(x+z) + 29(x+z)^2 \\ &= 20x^2 + 32xz + 29z^2, \end{aligned}$$

and then

$$\begin{gathered} 20x^2 + 32xz = 4(5x^2 + 8xz), \\ 5(5x^2 + 8xz) = (5x + 4z)^2 - 16z^2, \\ 5 \cdot 29 - 4 \cdot 16 = 145 - 64 = 81 = 9^2, \end{gathered}$$

and so (4) is equivalent to

$$\begin{gathered} 4(5x + 4z)^2 + (9z)^2 = 5 \cdot 18^2, \\ 4\big(5x + 4(y - x)\big)^2 + \big(9(y - x)\big)^2 = 5 \cdot 18^2, \\ 4(x + 4y)^2 + (9x - 9y)^2 = 5 \cdot 18^2, \\ \left(x\frac{2\sqrt{5}}{5} + y\frac{8\sqrt{5}}{5}\right)^2 + \left(x\frac{9\sqrt{5}}{5} - y\frac{9\sqrt{5}}{5}\right)^2 = 18^2. \end{gathered}$$

Thus conjugate axes are segments of the lines

$$x - y = 0, \qquad\qquad x + 4y = 0,$$

and corresponding conjugate axes are

$$\left(\frac{9\sqrt{5}}{5}, \frac{9\sqrt{5}}{5}\right), \qquad\qquad \left(\frac{8\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\right),$$

which are about $(4.02, 4.02)$ and $(3.58, -0.89)$, as in Figure 9.

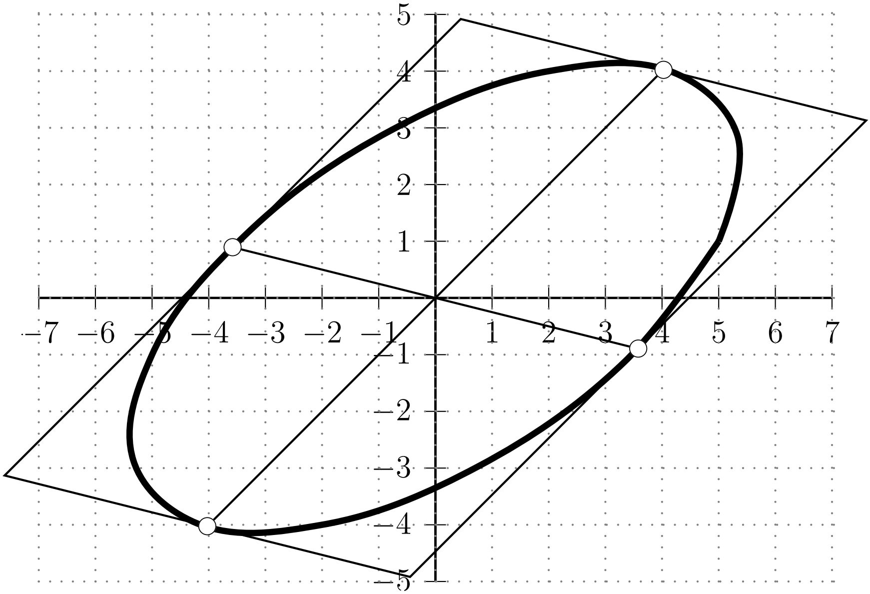

Figure 9: Ellipse with oblique diameter

12 Hyperbola

In place of (3), we can work with

$$(4x - 2y)^2 - (x - 5y)^2 = 18^2, \tag{17}$$

which defines the *hyperbola* shown in Figure 10. The same segments of of the lines defined by (1) are, by definition, conjugate diameters of the hyperbola, as they are of the ellipse. However, the endpoints of one of the diameters do not actually lie on the hyperbola. We can still find axes, as we did for the ellipse. First, in place of (4), we obtain the defining equation

$$15x^2 - 6xy - 21y^2 = 18^2,$$

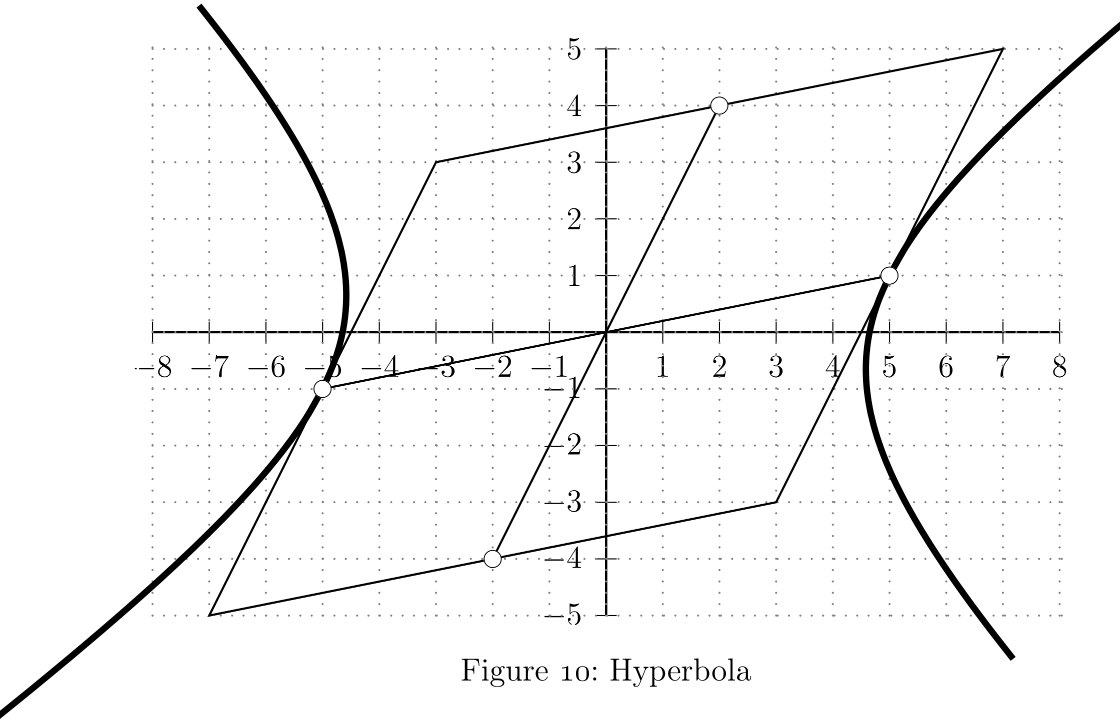

Figure 10: Hyperbola

which reduces to

$$5x^2 - 2xy - 7y^2 = 108, \tag{18}$$

though we shall have to remember that this is no longer a vertical equation. Now complete the squares. Since

$$\begin{aligned} 5(5x^2 - 2xy) &= (5x)^2 - 2(5x)y \\ &= (5x - y)^2 - y^2, \end{aligned}$$

the hyperbola defined by (17) and (18) is defined also by

$$(5x - y)^2 - (6y)^2 = 540,$$

with corresponding vertical equation

$$\left(x\sqrt{15} - y\frac{\sqrt{15}}{5}\right)^2 - \left(6y\frac{\sqrt{15}}{5}\right)^2 = 18^2,$$

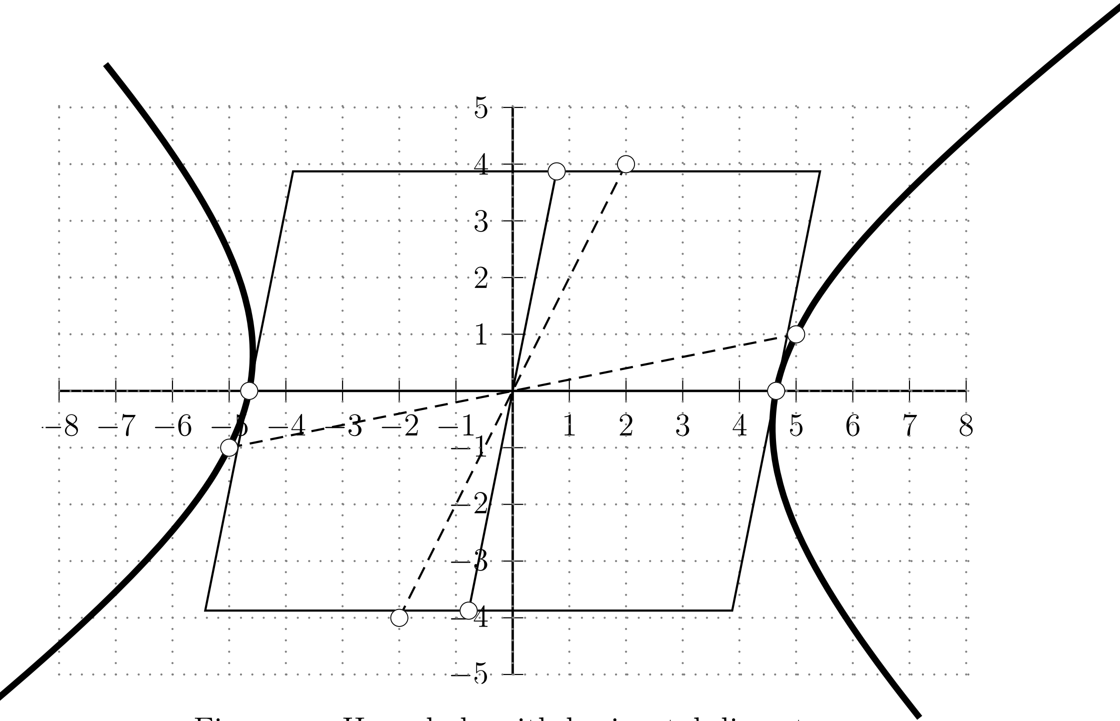

Figure 11: Hyperbola with horizontal diameter

so that conjugate vertices are

$$\left(\frac{6\sqrt{15}}{5}, 0\right), \qquad \left(\frac{\sqrt{15}}{5}, \sqrt{15}\right),$$

which are about $(4.64, 0)$ and $(0.77, 3.87)$ as in Figure 11. Towards finding the axes, we intersect with the circle defined by

$$(5x)^2 + (5y)^2 = 540,$$

as in Figure 12. Simultaneous solution yields

$$\begin{gathered}(5x - y)^2 - (6y)^2 = (5x)^2 + (5y)^2,\\ -10xy + (1 - 6^2 - 5^2)y^2 = 0,\\ y(x + 6y) = 0.\end{gathered}$$

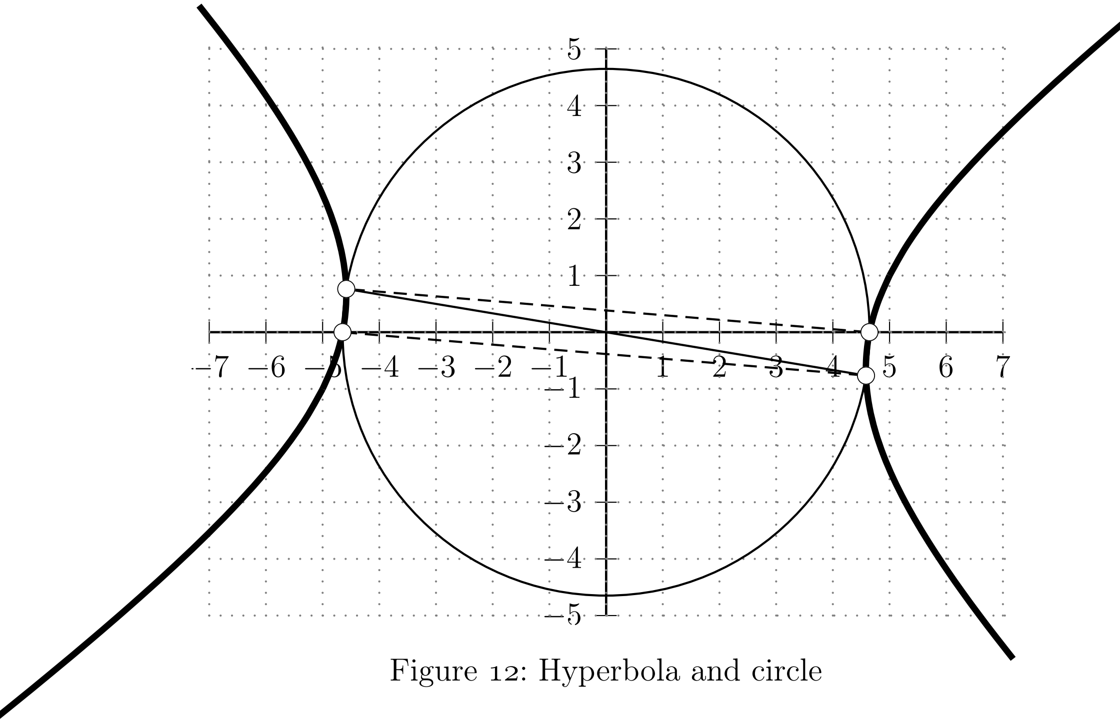

Figure 12: Hyperbola and circle

The line given by $x + 6y = 0$ meets the hyperbola and circle where

$$x = -6y,$$
$$15 \cdot 6^2 = 540 = (6^2 + 1)(5y)^2 = 37(5y)^2,$$
$$(x, y) = \left(\frac{\mp 36\sqrt{15}}{5\sqrt{37}}, \frac{\pm 6\sqrt{15}}{5\sqrt{37}} \right),$$

and these points are about $(\mp 4.58, \pm 0.76)$. We can now compute the slope of one axis:

$$\frac{\dfrac{-6\sqrt{15}}{5\sqrt{37}}}{\dfrac{6\sqrt{15}}{5} + \dfrac{36\sqrt{15}}{5\sqrt{37}}} = \frac{-1}{\sqrt{37} + 6}.$$

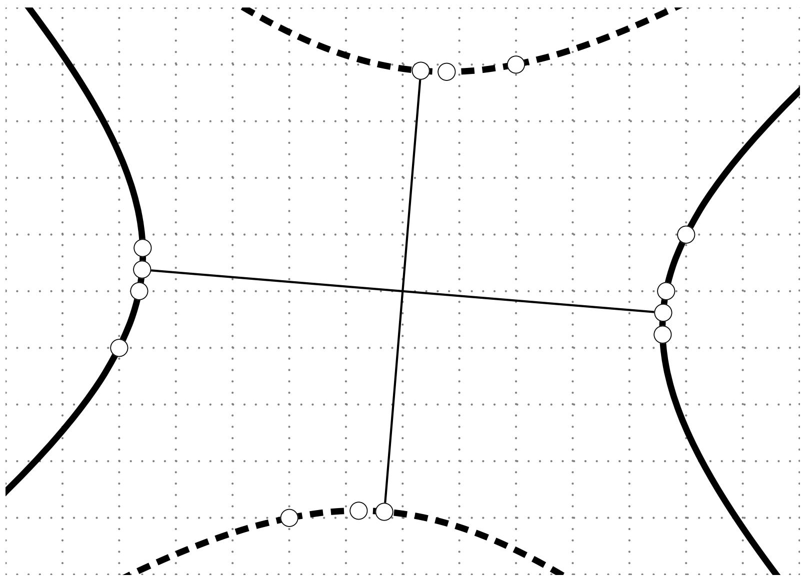

Figure 13: Conjugate hyperbolas and shared axes

Thus the axes are segments of the lines defined respectively by

$$y = (\sqrt{37} + 6)x, \qquad (\sqrt{37} + 6)y = -x;$$

these can be seen in Figure 13. We rewrite the last equations as

$$(6 + \sqrt{37})x - y = 0, \qquad x + (6 + \sqrt{37})y = 0.$$

Now we expect the hyperbola to be defined by an equation of the form

$$\alpha\big((\sqrt{37} + 6)x - y\big)^2 + \beta\big(x + (\sqrt{37} + 6)y\big)^2 = 108.$$

We verify this by defining

$$\gamma = 1 + (\sqrt{37} + 6)^2$$

$$
\begin{aligned}
&= 2 \cdot 37 + 2 \cdot 6\sqrt{37} \\
&= 2(\sqrt{37}+6)\sqrt{37},
\end{aligned}
$$

and now solving

$$
\begin{aligned}
(\gamma-1)\alpha + \qquad\quad \beta &= 5, \\
-\gamma\alpha + \qquad\quad \gamma\beta &= -2\sqrt{37}, \\
\alpha + (\gamma-1)\beta &= -7,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(\gamma-1)\alpha + \quad \beta &= 5, \\
-\gamma\alpha + \gamma\beta &= -2\sqrt{37}, \\
\gamma\alpha + \gamma\beta &= -2,
\end{aligned}
$$

and then to

$$
\begin{aligned}
(\gamma-1)\alpha + \quad \beta &= 5, \\
\gamma\beta &= -(\sqrt{37}+1), \\
\gamma\alpha \qquad\quad &= \sqrt{37}-1,
\end{aligned}
$$

which indeed has the obvious solution. The hyperbola is now defined by

$$
\frac{\sqrt{37}-1}{\gamma}\big((\sqrt{37}+6)x-y\big)^2 - \frac{\sqrt{37}+1}{\gamma}\big(x+(\sqrt{37}+6)y\big)^2 = 108.
$$

Since

$$
\begin{aligned}
\gamma(\sqrt{37}-6) &= 2\sqrt{37}, \\
(\sqrt{37}-1)(\sqrt{37}-6) &= 43-7\sqrt{37}, \\
(\sqrt{37}+1)(\sqrt{37}-6) &= 31-5\sqrt{37},
\end{aligned}
$$

the hyperbola is defined by

$$\frac{43-7\sqrt{37}}{2\sqrt{37}}\big((\sqrt{37}+6)x-y\big)^2 - \frac{31-5\sqrt{37}}{2\sqrt{37}}\big(x+(\sqrt{37}+6)y\big)^2 = 108,$$

hence by the vertical equation

$$\frac{3(43-7\sqrt{37})}{2\sqrt{37}}\big((\sqrt{37}+6)x-y\big)^2 - \frac{3(31-5\sqrt{37})}{2\sqrt{37}}\big(x+(\sqrt{37}+6)y\big)^2 = 18^2. \quad (19)$$

Corresponding conjugate vertices are

$$\pm\left((\sqrt{37}+6)\sqrt{\frac{3(31-5\sqrt{37})}{2\sqrt{37}}}, -\sqrt{\frac{3(31-5\sqrt{37})}{2\sqrt{37}}}\right),$$
$$\pm\left(\sqrt{\frac{3(43-7\sqrt{37})}{2\sqrt{37}}}, (\sqrt{37}+6)\sqrt{\frac{3(43-7\sqrt{37})}{2\sqrt{37}}}\right).$$

In (19), if we replace $\sqrt{37}$ with $-\sqrt{37}$, we obtain an equation for the *conjugate* hyperbola, which has the same axes and is shown in Figure 13; its equation is also (19) itself, but with the sign of the constant term 18^2 reversed. Thus our hyperbola itself is given also by

$$\frac{3(43+7\sqrt{37})}{2\sqrt{37}}\big((\sqrt{37}-6)x+y\big)^2 - \frac{3(31+5\sqrt{37})}{2\sqrt{37}}\big(x-(\sqrt{37}-6)y\big)^2 = 18^2,$$

and so it has conjugate vertices

$$\pm\left((\sqrt{37}-6)\sqrt{\frac{3(31+5\sqrt{37})}{2\sqrt{37}}}, \sqrt{\frac{3(31+5\sqrt{37})}{2\sqrt{37}}}\right),$$

$$\pm\left(-\sqrt{\frac{3(43+7\sqrt{37})}{2\sqrt{37}}}, (\sqrt{37}-6)\sqrt{\frac{3(43+7\sqrt{37})}{2\sqrt{37}}}\right).$$

Again, these are the same as the vertices already computed; comparing quadrants as before, we can write conjugate vertices as

$$\left(\sqrt{\frac{3(43\pm7\sqrt{37})}{2\sqrt{37}}}, \mp\sqrt{\frac{3(31\mp5\sqrt{37})}{2\sqrt{37}}}\right),$$

and (19) as

$$\left(x\sqrt{\frac{3(31+5\sqrt{37})}{2\sqrt{37}}} - y\sqrt{\frac{3(43-7\sqrt{37})}{2\sqrt{37}}}\right)^2 - \left(x\sqrt{\frac{3(31-5\sqrt{37})}{2\sqrt{37}}} + y\sqrt{\frac{3(43+7\sqrt{37})}{2\sqrt{37}}}\right)^2 = 18^2.$$

13 Determinants

In (3), the number 18 is the **determinant** of the 2×2 matrix whose top row is $(5,1)$ and whose bottom row is $(2,4)$. In short,

$$18 = \det\begin{pmatrix}5 & 1\\ 2 & 4\end{pmatrix}.$$

In general, by definition,

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc.$$

Then, by computation,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c+ta & d+tb\end{pmatrix}.$$

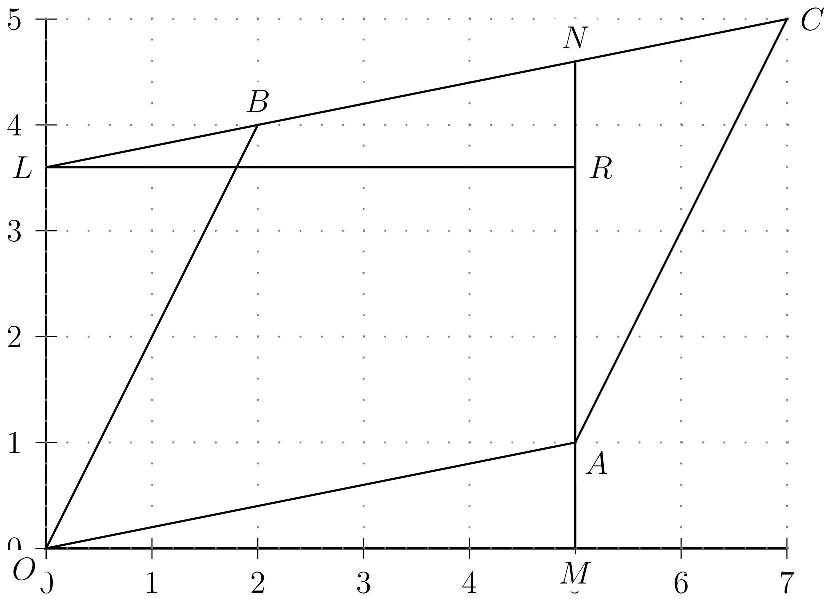

Figure 14: Determinant as area

If $a \neq 0$, then we may let $t = -c/a$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}.$$

Finally,

$$\begin{pmatrix} a & b \\ 0 & e \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}.$$

In our example, given points A and B as in Figure 14, we may slide B along the line parallel to OA until it reaches the y-axis as L, then slide A parallel to the y-axis until it reaches the x-axis as M. Three parallelograms are determined, all equal to one another:

$$OACB = OANL = OMRL.$$

Moreover, the last parallelogram is a rectangle. If the coordinates of A and B are (a, b) and (c, d), then the coordinates of L are $(0, d - bc/a)$, and the coordinates of M are $(1, 0)$. Thus the determinant $ad - bc$ is the area of the parallelogram $OACB$. The area is *signed*: interchanging the rows of the matrix changes the sign of the determinant. Using the abbreviation

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

we can write the vertical equation of the ellipse with center $(0, 0)$ and conjugate vertices (a_1, a_2) and (b_1, b_2) as

$$\begin{vmatrix} x & y \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ x & y \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2. \tag{20}$$

Let us now identify (a_1, a_2) with the **vector,** denoted by $\boldsymbol{a}$, representing the directed segment from the origin to (a_1, a_2); and likewise, (b_1, b_2) with $\boldsymbol{b}$. We can add vectors and scale them. In (20), the first determinant is unchanged if (x, y) slides, parallel to $\boldsymbol{b}$, so as to meet $\boldsymbol{a}$; and the meeting point will be the endpoint of a vector $s \cdot \boldsymbol{a}$. likewise for the second determinant, if (x, y) slides parallel to $\boldsymbol{a}$ so as to meet $\boldsymbol{b}$: the meeting point will be the endpoint of a vector $t \cdot \boldsymbol{b}$. Thus, as in Figure 15, the ellipse consists of the points $s \cdot \boldsymbol{a} + t \cdot \boldsymbol{b}$, where

$$s^2 + t^2 = 1.$$

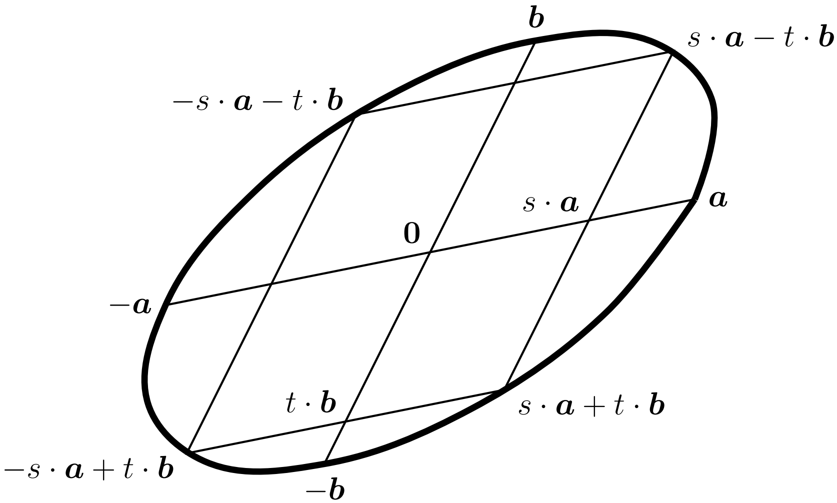

Figure 15: Ellipse