Notes from a talk at "Model Theory and Mathematical Logic: Conference in honor of Chris Laskowski's 60th birthday," University of Maryland, College Park

Ratio Then and Now

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1 Preface

The notes in §3 correspond roughly to what I said (or could have said) in the talk itself, except for some remarks then about the case of Tuna Altınel.

The talk fleshed out part 2 of the abstract (in §2), then gave an axiomatization of affine planes (with two sorts, for points and for polygons) in terms of *equality* of polygons (in the sense of Euclid: weaker than congruence, not to mention identity).

I now take up the remaining two parts of the abstract in §4.

I had presented some of the material in earlier talks (for which notes are in the given subdirectories of ⟨`mat.msgsu.edu.tr/~dpierce/Talks/`⟩) and a paper:

- "Euclid Mathematically and Historically," a 50-minute colloquium talk in the mathematics department of Bilkent University, Ankara, March 7, 2018 ⟨`2018-Bilkent/`⟩;
- "Conic Sections With and Without Algebra," a 20-minute contributed talk at Antalya Algebra Days, Nesin Mathematics Village, May 15, 2019 ⟨`2019-AAD/`⟩;
- "Thales and the Nine-point Conic," *The De Morgan Gazette,* 2016 [24].

After July 1, 2019, I plan to post the present notes and more ⟨`2019-UMd/`⟩.

2 Abstract

Having submitted this as plain text, I typeset it here as such.

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``Heraclitus holds that the findings of
sense-experience are untrustworthy, and he sets up
reason [logos, ratio] as the criterion'' (Sextus
Empiricus)

``It is necessary to know that war is common and
right is strife [eris] and all things happen by
strife and necessity'' (Heraclitus, according to
Origen)
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1. Strife has arisen between the historian of mathematics and the mathematician who thinks about the past. One must be both, to understand Euclid's obscure definition of proportion of numbers. Proportion is sameness of ratio. When this occurs between two pairs of numbers, something should be the same about each pair. In Book VII of the Elements, this can only mean that the Euclidean Algorithm has the same steps when applied to either pair of numbers. From this, despite modern suggestions to the contrary, Euclid has rigorous proofs, not only of what we call Euclid's Lemma, but also of the commutativity of multiplication.

2. Apollonius of Perga gives three ways to characterize a conic section: (i) an equation, involving a latus rectum, that we can express in Cartesian form; (ii) the proportion whereby the square on the ordinate varies as the abscissa or product of abscissas; (iii) an equation of a triangle with a parallelogram or trapezoid. The latter equation holds in an affine plane. With the advent of Cartesian methods in 1637, the equation seems to have been forgotten, because it is not readily translated into the lengths (symbolized by single minuscule letters) that Descartes has taught us to work with. With the affine equation, Apollonius can give a proof-without-words of what today we consider a coordinate change, performed with more or less laborious computations.

3. By interpreting the field where algebra is done

in the plane where geometry is done, Descartes does inspire new results. An example still builds on work of an ancient mathematician, Pappus of Alexandria. The model companion of the theory of Pappian affine spaces of unspecified dimension, considered as sets of points with ternary relation of collinearity and quaternary relation of parallelism, is the theory of Pappian affine planes over algebraically closed fields.

3 Lecture notes

In the title,

- "Ratio" is either the mathematical concept, or the Latin source of our word *reason*;
- "Then" and "Now" are divided by 1637, the year of publication of Descartes's *Geometry* [4, 5], the origin of the polynomial equations, such as

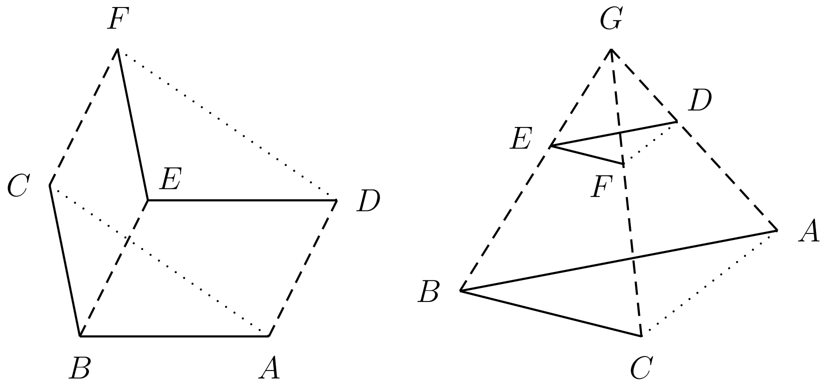
$$x^2 \pm y^2 = 1, \tag{1}$$

  that either frighten or excite us in high school today.

In such an equation as (1), the minuscule letters stand for *lengths,* and then so do their sums and products. There is then no bound on degree. However, the equations do not capture such ancient work as a forgotten proof by Apollonius [1, 2], which relies on *areas.*

Artin [3] shows how to coordinatize as K^2, for some field K, the **affine plane** axiomatized by:

1) two points determine a line;
2) **Playfair's axiom,** that through a point not on a line passes a unique parallel to that line;



(a) Parallel case (b) Intersecting case

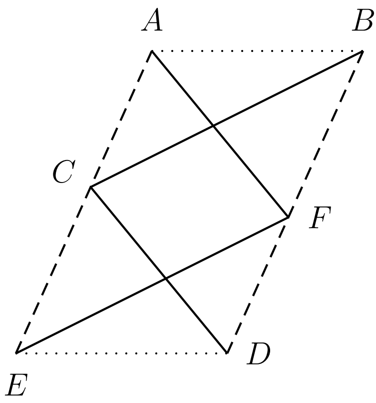
Figure 1: Desargues's Theorem

3) nontriviality (there are three non-collinear points);
4) **Desargues's Theorem,** that if the lines AD, BE, and CF either
   (a) are mutually parallel as in Fig. 1a, or
   (b) have a common point G as in Fig. 1b,

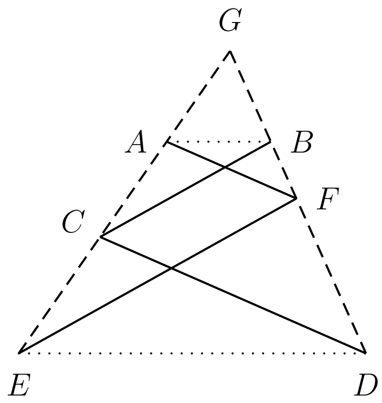
   then
   $$AB \parallel DE \;\&\; BC \parallel EF \Rightarrow CA \parallel FD;$$
5) **Pappus's Theorem,** that if A, C, and E are distinct collinear points as in Fig. 2, as are B, D, and F, then
   $$CD \parallel FA \;\&\; BC \parallel EF \Rightarrow AB \parallel DE.$$

By one argument (not exactly Artin's),

- by the parallel case of Desargues, the transitive closure of the relation of being opposite directed sides of a parallelogram is a definable equivalence relation whose classes



(a) Parallel case



(b) Intersecting case

Figure 2: Pappus's Theorem

we call **vectors,** and then, by Playfair,

$$\overrightarrow{AB} = \overrightarrow{AC} \Rightarrow B \equiv C;$$

- by the parallel case of Pappus, the vectors compose an abelian group;
- by the intersecting case of Desargues,
  - the transitive closure of the relation between ordered pairs of parallel vectors whose representives, when they share an initial point, have respective terminal points lying on parallel lines is a definable equivalence relation whose classes are **ratios,** and then, by Playfair,

    $$\overrightarrow{AC} : \overrightarrow{AB} :: \overrightarrow{AD} : \overrightarrow{AB} \Rightarrow C \equiv D;$$

  - the ratios compose a field, possibly not commutative;

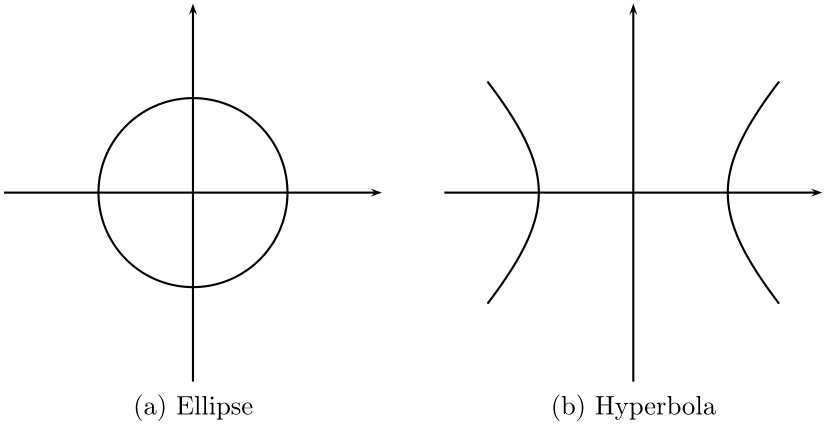


Figure 3: Ellipse and hyperbola

   – the field of ratios acts on the group of vectors, making this a two-dimensional vector space;
- by the intersecting case of Pappus, the field of ratios is commutative.

Pappus's Theorem was originally Lemma VIII of the lemmas for Euclid's now-lost *Porisms,* in Book VII of Pappus's *Collection* [18, 19, 20].[1] Enumerated carefully, the principles used in the proof will become axioms for affine planes with an additional sort for *polygons.* This will be a better setting for the Apollonian proof.

On the curve given by (1), as in Fig. 3, a point (a, b) is interchanged with $(1, 0)$ by an *affinity* or affine transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

[1] I did not distinguish the parallel case of Pappus in the talk, and Pappus does not prove it explicitly, though by his methods it has an easier proof than the intersecting case.

fixing $(0,0)$, where

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a^2+cb\\ ba+db\end{pmatrix},$$

so that, since

$$a^2 \pm b^2 = 1,$$

we compute

$$c = \pm b, \qquad\qquad d = -a,$$

and

$$\begin{pmatrix}a & \pm b\\ b & -a\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax \pm by\\ bx - ay\end{pmatrix}.$$

Since

$$(ax \pm by)^2 \pm (bx - ay)^2 = x^2 \pm y^2,$$

the image $(ax \pm by, bx - ay)$ lies on our curve if and only if (x, y) does. That's the cleanest modern, Cartesian proof I can come up with for the theorem that the affinity fixes the curve.

The proof seems to have taken centuries of development since 1637. The earliest version, by William Wallis in 1655 [28], involves many letters and intricate computations.

As a corollary, for arbitrary V^* on the locus of P, where

$$\overrightarrow{KP} = x \cdot \overrightarrow{KV} + y \cdot \overrightarrow{KL},$$

where again (1) holds, and $\overrightarrow{KV}$ and $\overrightarrow{KL}$ are independent vectors as in Fig. 4 or Fig. 5, the affinity that fixes K and interchanges V and V^* fixes the curve. For Apollonius's proof, we draw VE^*, V^*M, and PX as shown, parallel to KL. Then

$$x = \overrightarrow{KX} : \overrightarrow{KV}, \qquad\qquad y = \overrightarrow{XP} : \overrightarrow{KL}.$$

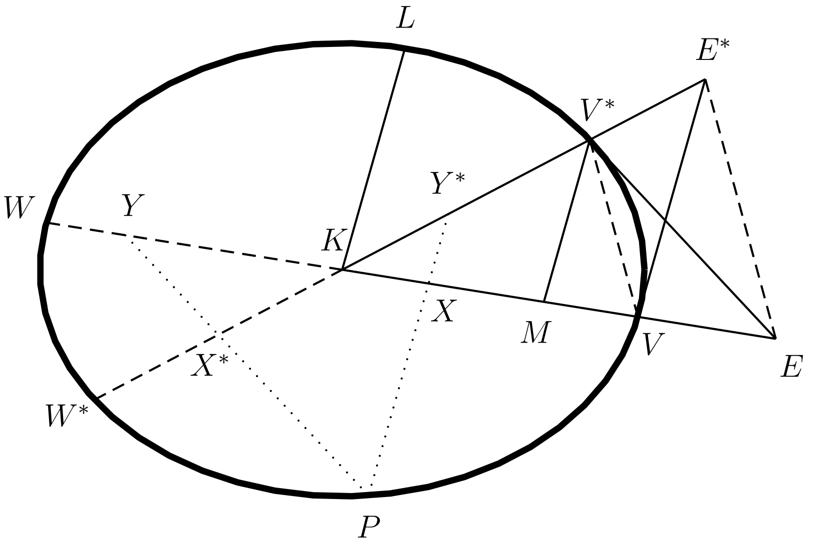


Figure 4: Ellipse

Writing (1) as

$$\pm y^2 = (1-x)(1+x),$$

we obtain from this

$$\overrightarrow{XP}^2 \propto \overrightarrow{XV} \cdot \overrightarrow{WX}.$$

Since

$$\overrightarrow{WX} = \overrightarrow{KX} + \overrightarrow{KV} \propto \overrightarrow{XY^*} + \overrightarrow{VE^*},$$

we conclude

$$\overrightarrow{XP}^2 \propto VXY^*E^*.$$

Letting $V^*V \parallel E^*E$ yields

$$MV^*E = VMV^*E^*. \tag{2}$$

Dropping PY to KV parallel to V^*E, we have

$$\overrightarrow{KP}^2 \propto XPY,$$

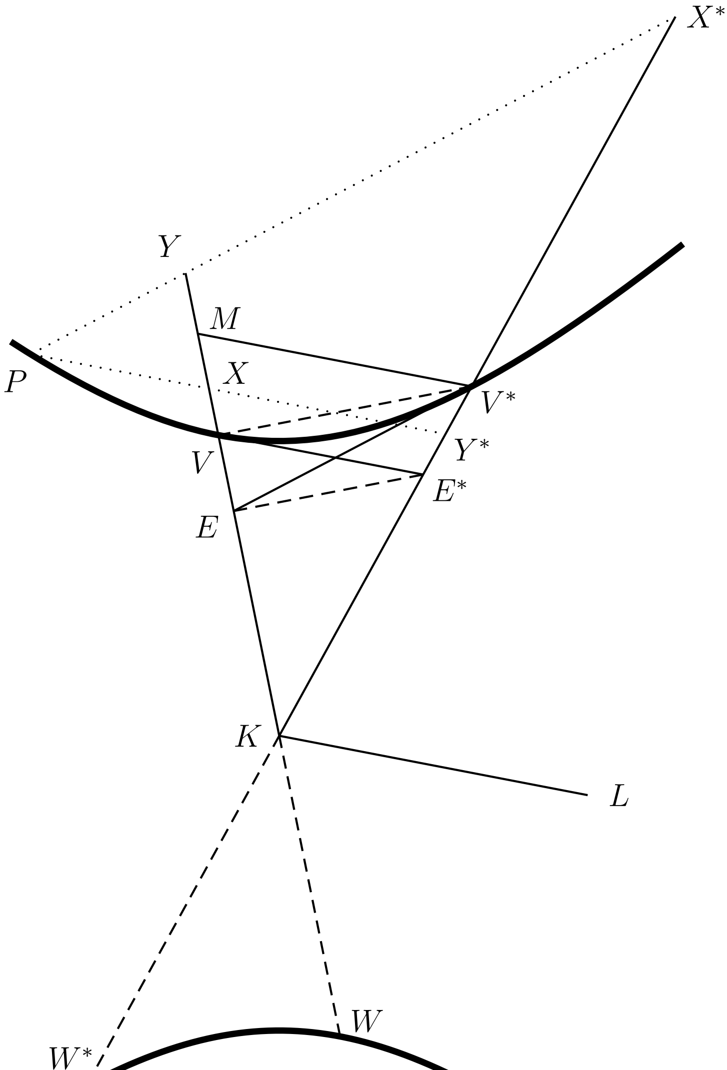


Figure 5: Hyperbola

and therefore

$$XPY \propto VXY^*E^*.$$

Since this becomes an equation, namely (2), when P is V^*, we conclude

$$XPY = VXY^*E^*. \tag{3}$$

This is an alternative defining equation for our curve. If to either side we add the quadrilateral

$$YX^*Y^*X,$$

we obtain

$$Y^*PX^* = VYX^*E^*.$$

Applying (2) again, we conclude

$$Y^*PX^* = EYX^*V^*.$$

This is (3) with reversed orientation, with respect to a new basis. That is the proof of Apollonius. It is remarked on, neither by the mathematician Rosenfeld [25, p. 57] nor the mathematical historians Fried and Unguru [13].

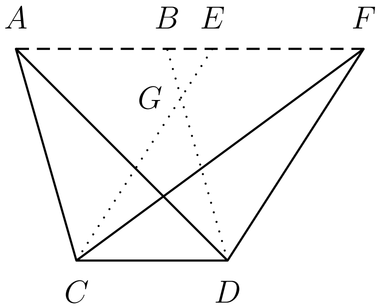
The proof, and specifically (2), relies on Propositions 37 and 39 of Euclid's *Elements* [6, 7, 9, 10, 11]: in Fig. 6,

$$AF \parallel CD \Leftrightarrow ACD = FCD.$$

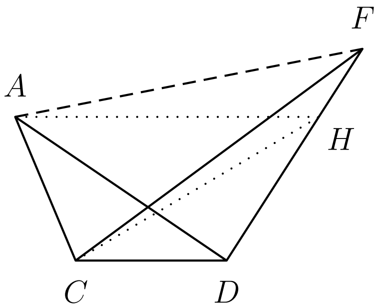
We can understand Euclid's proof as having the following steps, the justifications of which will be sufficient to axiomatize an affine plane of null or odd characteristic.

1. First, assuming $AF \parallel CD$, we let

$$AC \parallel BD, \qquad\qquad CE \parallel DF.$$



(a) Proposition 37



(b) Proposition 39

Figure 6: Parallelism

2. Then
$$ACE = BDF.$$

3. We can now compute
$$\begin{aligned} ACDB &= ACGB + CDG \\ &= ACE - BGE + CDG \\ &= BDF - BGE + CDG \\ &= ECDF. \end{aligned}$$

4. Thus
$$2ACD = 2FCD.$$

5. Hence
$$ACD = FCD.$$

6. Now supposing $AF \nparallel CD$, we let $AH \parallel CD$, concluding
$$ACD = HCD,$$
$$HCD + FCH = FCD,$$

and therefore

$$ACD \neq FCD$$

since $FCH \neq 0$.

The corresponding axioms are:

6. A, B, and C are collinear if and only if $ABC = 0$.
1. Playfair's Axiom.
3. The polygons compose an abelian group, according to the following rules, where Γ and Δ are strings of letters for points:

$$A\Gamma = A\Gamma A = \Gamma A,$$
$$A\Gamma B + B\Delta A = A\Gamma B\Delta,$$
$$-A_1 \cdots A_n = A_n \cdots A_1.$$

The group either is torsion-free or is a vector-space over a field of prime order p.

5. $p \neq 2$.
4. Diagonals bisect parallelograms.
2. Side-Angle-Side: In Fig. 1a,

$$AB \parallel DE \ \&\ BC \parallel EF \Rightarrow ABC = DEF.$$

Now we can apply I.39 to the situation of Axiom 2, obtaining the parallel case of Desargues's Theorem. Also I.37 and 39 together yield Pappus's Theorem.

Selecting now a proper triangle IOI', we can define a multiplication on OI as in Fig. 7, then show it to be commutative and associative.

To ensure that the multiplication is independent of I', we establish Desargues in the case shown in Fig. 8a (and worked out in [24]), where

$$AC \parallel BD \ \&\ AE \parallel BF \Rightarrow EC \parallel FD$$

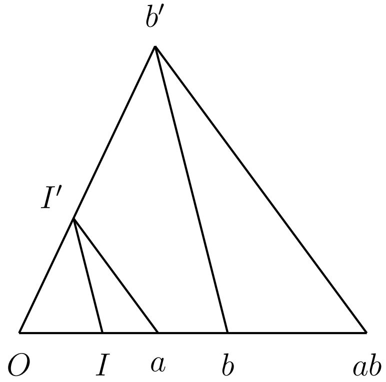


Figure 7: Multiplication

under the assumption

$$AE \parallel OC.$$

The proof uses an embellishment of Euclid's Proposition I.43 and its converse, that in Fig. 8b,

$$AC \parallel BD \iff AGNB = GCDM \iff OGL = 0.$$

In Fig. 8a now, assuming

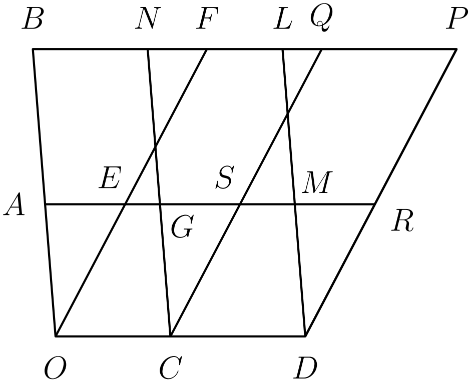
$$AC \parallel BD, \qquad AE \parallel BF \parallel OC,$$

we have

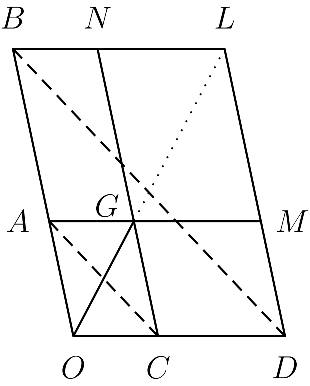
$$CDRS = CDMG = AGNB = ESQF,$$

so

$$EC \parallel FD.$$



(a) Special case of Desargues



(b) Euclid I.43

Figure 8

4 Epilogue

Part 3 of the abstract alludes to a straightforward corollary to work in [23]. John Baldwin has referred me to related work of Manders [15].

As for Part 1 of the abstract, in Book VII of Euclid's *Elements,* we are told about numbers that, if A is the same *part,* or *parts,* or *multiple,* of B that C is of D, then

- A, B, C, and D are **proportional,** in the language of Definition 20,
- A is to B **as** C is to D, in the language of Proposition 11;
- A has the **same ratio** to B that C has to D, in the language of Proposition 17.

Moreover, the following are equivalent for numbers A and B:

- A is **part** of B;
- B is a **multiple** of A;

- A **measures** B.

Today, instead of measuring, we refer to *dividing;* but they are the same, only because multiplication of two numbers is commutative; and Euclid proves this.

If it is not *part* of the greater, the less number is **parts** of the greater. So-called Definition 4 says this, but then so does Proposition 4, while having a nontrivial proof, establishing the meaning of "same parts," as Zeuthen [29, p. 410] and Itard [14, p. 90] observe.

By the first three propositions of Book VII,

1) if the Euclidean Algorithm, applied to two numbers, yields unity, the numbers are co-prime;
2) if they are not co-prime, the Algorithm yields their greatest common measure;
3) we can obtain the greatest common measure of three numbers by the Algorithm as well.

Now suppose $A < B$. According to the proof of Proposition 4,

1. If A and B are co-prime, then each unit of A is part of B, so A is parts of B.
2. If A and B are not co-prime, but A measures B, then A is part of B.
3. In the remaining case, let C be the greatest common measure of A and B. Then A is the sum of parts, each equal to C and thus to a part of B, and therefore A is parts of B.

The rule must be that if, for some C and multipliers k and m, where $k < m$,

$$A = k \cdot C, \qquad\qquad B = m \cdot C,$$

then

- A is **part** of B if $k = 1$,

- A is **parts** of B if $k > 1$,

—*provided* also C is the greatest common measure of A and B, or equivalently k and m are co-prime. Now the meaning of having the same ratio is clear: if also

$$D = k \cdot F, \qquad\qquad E = m \cdot F,$$

then D has the same ratio to E that A has to B. Having the same ratio is thus *obviously* a transitive relation. It would not be so, without the proviso that k and m be co-prime. Therefore, I say, Euclid cannot have meant us to disregard this proviso.

That he *did* mean us to disregard it is nonetheless the view of some researchers: I am aware of Mazur [16], Mueller [17, p. 62], Pengelley [21, p. 870], him and Richman [22, p. 199], Taisbak [26, pp. 30–2], and Vitrac [8, p. 300]. In this case, Euclid would be guilty of circularity or logical gaps.

There is little reason to suspect him of this, but more reason to think that his definition of proportion of numbers is a practical adaptation of an earlier definition (surmised by Becker in 1933 [27, pp. 504–9] and investigated most fully by Fowler [12], though not for numbers), whereby A is to B as C is to D, provided the Euclidean algorithm has the same steps, whether applied to A and B or to C and D.

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