Classification of regular polytopes

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Preface

For reasons I do not specifically recall, I first drafted this document in 2011. My interest did not last long enough for me to fill all gaps. During the coronavirus pandemic of 2020, again for unremembered reasons, I returned to the work and brought it to what seemed like a reasonable completion.

Contents

List of Tables

List of Figures

1 Introduction

The regular polytopes are the generalizations to arbitrary dimension of the regular polygons and the regular polyhedra. The latter are the five so-called Platonic solids. Among them, the tetrahedron, cube, and octahedron have versions in every dimension; the icosahedron and dodecahedron, only in the fourth dimension. There is one other regular polytope, also in the fourth dimension, and that is all.

We shall construct all of the regular polytopes by giving coordinates for their vertices in the appropriate space $\mathbb{R}^n$. For notational convenience,

$$n = \{i \colon i < n\} = \{0, \dots, n-1\};$$

in particular, n is a set with n elements. The same element of $\mathbb{R}^n$ can be written as any of

$$\boldsymbol{a}, \qquad (a_0, \dots, a_{n-1}), \qquad (a^0, \dots, a^{n-1}).$$

Proving that some points are the vertices of a regular polytope is ultimately a matter of showing that

- certain sets of them are coplanar;
- the distances between certain pairs of them are all the same.

A plane in $\mathbb{R}^n$ is defined by an equation

$$\sum_{i<n} a_i x^i = b,$$

where $\boldsymbol{a} \neq \mathbf{0}$. In $\mathbb{R}^n$, the distance $|\boldsymbol{a}|$ between $\boldsymbol{a}$ and $\mathbf{0}$ is

$$\sqrt{\sum_{i<n} {a_i}^2}.$$

Proofs involving a fourth or higher dimension do not require us to have intuition for that dimension. Nonetheless, the appeal of the mathematics of that dimension may be the possibility of developing an intuition for it.

That there are no more regular polytopes than the ones that we shall construct is shown by a generalization of Euclid's proof, at the end of the *Elements,* that there are no more regular polyhedra than the known five.

2 Polytopes

If some finite number of points in some $\mathbb{R}^n$ compose a set A, we let

$$\mathrm{H}(A) = \left\{ \sum_{\boldsymbol{x}\in A} t^{\boldsymbol{x}}\boldsymbol{x} \colon \bigwedge_{\boldsymbol{x}\in A} 0 \leqslant t^{\boldsymbol{x}} \wedge \sum_{\boldsymbol{x}\in A} t^{\boldsymbol{x}} = 1 \right\}.$$

Here the superscripts are indices; for I am trying to use a version of "Einstein notation," as described in the Wikipedia article of that name. If one wants indices to be numbers, then one can understand A as the range of a fuction $i \mapsto \boldsymbol{a}_i$ from some set m, that is, $\{0, \dots, m-1\}$, into $\mathbb{R}^n$, and then one can write

$$\mathrm{H}(A) = \left\{ \sum_{i<m} t^i\boldsymbol{a}_i \colon \bigwedge_{i<m} 0 \leqslant t^i \wedge \sum_{i<m} t^i = 1 \right\}.$$

In any case, $\mathrm{H}(A)$ is the set of **convex combinations** of elements of A and is therefore the **convex hull** of A. As the convex hull of some finite set, $\mathrm{H}(A)$ is a **polytope.** All of this terminology is found in Rockafellar, *Convex Analysis* [6, pp. 11–2].

Rockafellar defines also *vertex,* but in a more specialized sense than I propose to give. Let us say that the elements of A are **vertices** of $\mathrm{H}(A)$ and that they **span** $\mathrm{H}(A)$. In this sense, $\mathrm{H}(A)$ need not determine its vertices, any more than a group need determine its generators. However, a polytope has

Table 2.1: Some families of polytopes

The polytope called	is the convex hull of
$(n-1)$-**simplex**	$\{\mathbf{e}_k \colon k \in n\}$
n-**orthoplex**	$\{(-1)^i\mathbf{e}_k \colon (i,k) \in 2 \times n\}$
n-**cube**	$\left\{\sum_{k\in X} \mathbf{e}_k \colon X \subseteq n\right\}$

a unique minimal spanning set: this may be intuitively clear, but in any case we shall give a proof. The elements of that unique minimal spanning set are vertices in the strictest sense.

Some standard examples of polytopes and their vertices are assembled in Table 2.1, where

$$\mathbf{e}_k = (\mathrm{e}_k^i \colon i < n), \qquad \mathrm{e}_k^i = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k, \end{cases}$$

so that $(\mathbf{e}_k \colon k < n)$ is the standard basis of $\mathbb{R}^n$ as a vector space. The notation does not specify n. The given spanning sets are minimal and have certain symmetries. The 3-simplex is the **tetrahedron;** the 3-orthoplex, the **octahedron;** the 3-cube, the **cube.** These are three of the *Platonic solids,* the others being the *icosahedron* and the *dodecahedron*; we shall look at those later.

Two polytopes are **equivalent** if one of them is isometric to a dilatation of the other. We generally identify equivalent polytopes.

A polytope is n-**dimensional,** or is an n-**polytope,** if it is equivalent to a polytope in $\mathbb{R}^n$, but not $\mathbb{R}^{n-1}$. Thus

- a 0-polytope is a singleton;

- a 1-polytope is a line segment;
- a 2-polytope is a polygon;
- a 3-polytope is a polyhedron;
- a 4-polytope is a **polychoron.**[1]

The n-simplex, -orthoplex, and -cube are n-polytopes.

[1]Wikipedia (accessed March 11, 2010) attributes the name *polychoron* to Norman Johnson (a student of Coxeter) and George Olshevski. The Greek source for the latter half of the name is apparently not *χορός* 'dance, chorus,' but *χῶρος* or *χώρα* 'place.' The last word, in the sense of 'place in the country,' was the name of the Chora Church, outside the walls of what was then Constantinople. The church was made a mosque after the Ottoman conquest, then a museum under the Turkish Republic; it was reverted to a mosque in 2020.

3 Simplices

We may prefer to define the n-simplex, not in $\mathbb{R}^{n+1}$, but in $\mathbb{R}^n$. We give it the vertices $\boldsymbol{a}_0, \dots, \boldsymbol{a}_n$, as defined in two cases.[1]

3.1 Even dimensions

When $n = 2m$, we let

$$\boldsymbol{a}_i = \sum_{k<m} \left(\cos \frac{2\pi i(k+1)}{n+1} \mathbf{e}_{2k} + \sin \frac{2\pi i(k+1)}{n+1} \mathbf{e}_{2k+1} \right).$$

Since

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 2 - 2\cos(\alpha - \beta),$$

we have, if $i \neq j$,

$$|\boldsymbol{a}_i - \boldsymbol{a}_j|^2 = n - \sum_{k<m} 2\cos\frac{2\pi(i-j)(k+1)}{n+1}.$$

Moreover, $\cos\alpha = \cos(2\pi - \alpha)$ generally, and in particular

$$\cos\frac{2\pi(i-j)(k+1)}{n+1} = \cos\frac{2\pi(i-j)(n-k)}{n+1},$$

[1] The idea for the realization developed here comes from the Wikipedia article "Simplex," which, however, until I corrected it on September 30, 2020, gave the vertices of the 3-simplex with ± 1 in place of $\pm 1/\sqrt{2}$.

so that

$$\sum_{k<m} 2\cos\frac{2\pi(i-j)(k+1)}{n+1}$$
$$= \sum_{k<m}\left(\cos\frac{2\pi(i-j)(k+1)}{n+1} + \cos\frac{2\pi(i-j)(n-k)}{n+1}\right)$$
$$= \sum_{k<n}\cos\frac{2\pi(i-j)(k+1)}{n+1}.$$

Since by symmetry

$$\sum_{k<n+1}\cos\frac{2\pi(i-j)(k+1)}{n+1} = 0,$$

we conclude

$$|\boldsymbol{a}_i - \boldsymbol{a}_j|^2 = n+1.$$

3.2 Odd dimensions

When $n = 2m+1$, we let

$$\boldsymbol{a}_i = \sum_{k<m}\left(\cos\frac{\pi i(k+1)}{m+1}\mathbf{e}_{2k} + \sin\frac{\pi i(k+1)}{m+1}\mathbf{e}_{2k+1}\right)$$
$$+ \frac{(-1)^i}{\sqrt{2}}\mathbf{e}_{2m}.$$

So that computations as in the even case will go through, we may think

$$(-1)^i\mathbf{e}_{2m} = \cos(\pi i)\mathbf{e}_{2m} + \sin(\pi i)\mathbf{e}_n,$$

though we assign no value to $\mathbf{e}_n$. Again if $i \neq j$,

$$\begin{aligned} |\boldsymbol{a}_i - \boldsymbol{a}_j|^2 &= n - \sum_{k<m} 2\cos\frac{\pi(i-j)(k+1)}{m+1} - \cos\bigl(\pi(i-j)\bigr) \\ &= n - \sum_{k<n} \cos\frac{\pi(i-j)(k+1)}{m+1} \\ &= n + 1. \end{aligned}$$

3.3 Examples

1. In $\mathbb{R}$ the 1-simplex has vertices $\sqrt{2}/2$ and $-\sqrt{2}/2$, which are $\sqrt{2}$ from one another.
2. In $\mathbb{R}^2$ the 2-simplex has vertices

$$(1,0), \qquad \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \qquad \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right),$$

   which are $\sqrt{3}$ from one another.
3. In $\mathbb{R}^3$ the 3-simplex has vertices

$$\begin{aligned} &\left(1, 0, \frac{\sqrt{2}}{2}\right), && \left(0, 1, \frac{-\sqrt{2}}{2}\right), \\ &\left(-1, 0, \frac{\sqrt{2}}{2}\right), && \left(0, -1, \frac{-\sqrt{2}}{2}\right), \end{aligned}$$

   which are 2 from one another.

4. In $\mathbb{R}^4$ the 4-simplex has vertices

$$
\begin{gathered}
(1,0,1,0), \\
\left(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}\right), \\
\left(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}, \cos\frac{8\pi}{5}, \sin\frac{8\pi}{5}\right), \\
\left(\cos\frac{6\pi}{5}, \sin\frac{6\pi}{5}, \cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}\right), \\
\left(\cos\frac{8\pi}{5}, \sin\frac{8\pi}{5}, \cos\frac{6\pi}{5}, \sin\frac{6\pi}{5}\right),
\end{gathered}
$$

which are $\sqrt{5}$ from one another.

4 Convex sets

We examine the properties of the operation of forming convex hulls. We don't necessarily need to know these properties, depending on how much we want to be true of regular polytopes by definition.

For finite sets A and B of points of $\mathbb{R}^n$, it is immediate from the definition of a convex hull that

$$A \subseteq B \implies \mathrm{H}(A) \subseteq \mathrm{H}(B).$$

Now we can define the convex hull of an arbitrary set of points in $\mathbb{R}^n$: it is the union of the convex hulls of the finite subsets. Thus

$$\mathrm{H}(A) = \bigcup\{\mathrm{H}(X) : X \subseteq A \wedge |X| < \omega\}. \tag{4.1}$$

A **convex set** is the convex hull of some set. If A is given as $\{\dots\}$, then we can write $\mathrm{H}(A)$ as

$$\mathrm{H}(\dots).$$

Lemma 1. *For all* $\boldsymbol{a}$ *in* $\mathbb{R}^n$,

$$\mathrm{H}(\boldsymbol{a}) = \{\boldsymbol{a}\}.$$

Proof. Immediate from the definition of the convex hull of a finite set. □

Lemma 2. *For all subsets* A *of* $\mathbb{R}^n$,

$$A \subseteq \mathrm{H}(A).$$

Proof. For every $\boldsymbol{b}$ in A, by Lemma 1 and the definition (4.1),

$$\{\boldsymbol{b}\} = \mathrm{H}(\boldsymbol{b}), \qquad \mathrm{H}(\boldsymbol{b}) \subseteq \mathrm{H}(A). \qquad \square$$

Lemma 3. *For all subsets A and B of $\mathbb{R}^n$,*

$$A \subseteq B \implies \mathrm{H}(A) \subseteq \mathrm{H}(B).$$

Proof. Immediate from the definition (4.1). $\square$

Lemma 4. *For all subsets A of $\mathbb{R}^n$,*

$$\mathrm{H}(\mathrm{H}(A)) = \mathrm{H}(A).$$

Proof. We may assume A is finite. Suppose B is a finite subset of $\mathrm{H}(A)$, and $\boldsymbol{c} \in \mathrm{H}(B)$. Then

$$\boldsymbol{c} = \sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}}\boldsymbol{y}$$

for some $t^{\boldsymbol{y}}$, where

$$\bigwedge_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} \geqslant 0, \qquad \sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} = 1.$$

For each $\boldsymbol{b}$ in B,

$$\boldsymbol{b} = \sum_{\boldsymbol{x}\in A} s_{\boldsymbol{b}}^{\boldsymbol{x}}\boldsymbol{x}$$

for some $s_{\boldsymbol{b}}^{\boldsymbol{x}}$, where

$$\bigwedge_{\boldsymbol{x}\in A} s_{\boldsymbol{b}}^{\boldsymbol{x}} \geqslant 0, \qquad \sum_{\boldsymbol{x}\in A} s_{\boldsymbol{b}}^{\boldsymbol{x}} = 1.$$

Then

$$\boldsymbol{c} = \sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} \sum_{\boldsymbol{x}\in A} s_{\boldsymbol{y}}^{\boldsymbol{x}}\boldsymbol{x} = \sum_{\boldsymbol{x}\in A}\sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} s_{\boldsymbol{y}}^{\boldsymbol{x}}\boldsymbol{x},$$

$$\sum_{\boldsymbol{x}\in A}\sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} s_{\boldsymbol{y}}^{\boldsymbol{x}} = \sum_{\boldsymbol{y}\in B} t^{\boldsymbol{y}} \sum_{\boldsymbol{x}\in A} s_{\boldsymbol{y}}^{\boldsymbol{x}} = 1.$$

Thus $\boldsymbol{c} \in \mathrm{H}(A)$. $\square$

Theorem 1. *The convex sets are precisely the sets that are the convex hulls of themselves.*

Proof. 1. If $A = \mathrm{H}(A)$, then $A = \mathrm{H}(B)$ for some B, so A is a convex set.

2. Conversely, if A is a convex set, so that $A = \mathrm{H}(B)$ for some B, then by Lemma 4,

$$\mathrm{H}(A) = \mathrm{H}(\mathrm{H}(B)) = \mathrm{H}(B) = A. \qquad \square$$

By Lemmas 2, 3, and 4, whereby the function $X \mapsto \mathrm{H}(X)$ is *extensive, monotone,* and *idempotent,* the function is by definition a *closure operator* or *hull operator.* By Lemma 1 in addition, it would be the *closure map* of a *geometry,* if it satisfied the *exchange* property, namely

$$\boldsymbol{c} \in \mathrm{H}(A \cup \{\boldsymbol{b}\}) \land \boldsymbol{c} \notin \mathrm{H}(A) \implies \boldsymbol{b} \in \mathrm{H}(A \cup \{\boldsymbol{c}\}).$$

It has instead the *anti-exchange* property, namely

$$\boldsymbol{c} \in \mathrm{H}(A \cup \{\boldsymbol{b}\}) \land \boldsymbol{c} \notin \mathrm{H}(A) \cup \{\boldsymbol{b}\} \implies \boldsymbol{b} \notin \mathrm{H}(A \cup \{\boldsymbol{c}\}),$$

which will follow from Lemma 7 below.[1]

Lemma 5. *For all* $\boldsymbol{c}$ *and* A *and* B*,*

$$\boldsymbol{c} \in \mathrm{H}(A \cup B) \smallsetminus \mathrm{H}(A) \implies \exists \boldsymbol{x}\ \big(\boldsymbol{x} \in \mathrm{H}(B) \land \boldsymbol{c} \in \mathrm{H}(A \cup \{\boldsymbol{x}\})\big).$$

[1] The terminology in this paragraph is found in the Wikipedia articles "Closure operator," "Pregeometry (model theory)," and "Antimatroid," accessed September 28, 2020. The definitions of anti-exchange in the first and last of these articles were wrong until I corrected them.

Proof. We may assume $A \cap B = \varnothing$. We have

$$\boldsymbol{c} = \sum_{\boldsymbol{x} \in A \cup B} t^{\boldsymbol{x}} \boldsymbol{x}$$

for some $t^{\boldsymbol{x}}$, where

$$\bigwedge_{\boldsymbol{x} \in A \cup B} t^{\boldsymbol{x}} \geqslant 0, \qquad \sum_{\boldsymbol{x} \in A \cup B} t^{\boldsymbol{x}} = 1.$$

We let

$$t = \sum_{\boldsymbol{x} \in B} t^{\boldsymbol{x}}.$$

Then $t \neq 0$, since $\boldsymbol{c} \notin \mathrm{H}(A)$. Consequently

$$\sum_{\boldsymbol{x} \in B} \frac{t^{\boldsymbol{x}}}{t} \boldsymbol{x} \in \mathrm{H}(B), \qquad \boldsymbol{c} = \sum_{\boldsymbol{x} \in A} t^{\boldsymbol{x}} \boldsymbol{x} + t \sum_{\boldsymbol{x} \in B} \frac{t^{\boldsymbol{x}}}{t} \boldsymbol{x}. \qquad \square$$

Lemma 6. *If* $\boldsymbol{c} \in \mathrm{H}(\boldsymbol{a}, \boldsymbol{b})$, *then*

$$\mathrm{H}(\boldsymbol{a}, \boldsymbol{b}) = \mathrm{H}(\boldsymbol{a}, \boldsymbol{c}) \cup \mathrm{H}(\boldsymbol{c}, \boldsymbol{b}).$$

Proof. We have

$$\boldsymbol{c} = (1-t)\boldsymbol{a} + t\boldsymbol{b}, \qquad 0 \leqslant t \leqslant 1,$$

for some t. Suppose $\boldsymbol{d} \in \mathrm{H}(\boldsymbol{a}, \boldsymbol{b})$. Then

$$\boldsymbol{d} = (1-u)\boldsymbol{a} + u\boldsymbol{b}, \qquad 0 \leqslant u \leqslant 1$$

for some u. We may assume $0 < u \leqslant t < 1$. Then $\boldsymbol{d} \in \mathrm{H}(\boldsymbol{a}, \boldsymbol{c})$, since

$$\boldsymbol{d} = (1-u)\boldsymbol{a} + \frac{u}{t}(\boldsymbol{c} - (1-t)\boldsymbol{a}) = \left(1 - \frac{u}{t}\right)\boldsymbol{a} + \frac{u}{t}\boldsymbol{c}. \qquad \square$$

Lemma 7. *If*

$$\boldsymbol{b} \notin \mathrm{H}\,(A)\,, \qquad\qquad \mathrm{H}\,(A \cup \{\boldsymbol{b}\}) = \mathrm{H}\,(A \cup \{\boldsymbol{c}\})\,,$$

then

$$\boldsymbol{b} = \boldsymbol{c}.$$

Proof. Under the hypothesis,

$$\boldsymbol{b} = \sum_{\boldsymbol{x} \in A} s^{\boldsymbol{x}} \boldsymbol{x} + s\boldsymbol{c}, \qquad\qquad \boldsymbol{c} = \sum_{\boldsymbol{x} \in A} t^{\boldsymbol{x}} \boldsymbol{x} + t\boldsymbol{b}$$

for some $s^{\boldsymbol{x}}$, s, $t^{\boldsymbol{x}}$, and t such that

$$\bigwedge_{\boldsymbol{x} \in A} (s^{\boldsymbol{x}} \geqslant 0 \wedge t^{\boldsymbol{x}} \geqslant 0) \wedge s \geqslant 0 \wedge t \geqslant 0,$$

$$\sum_{\boldsymbol{x} \in A} s^{\boldsymbol{x}} + s = 1 \wedge \sum_{\boldsymbol{x} \in A} t^{\boldsymbol{x}} + t = 1.$$

Then

$$\boldsymbol{b} = \sum_{\boldsymbol{a}} (s^{\boldsymbol{a}} + st^{\boldsymbol{a}})\boldsymbol{a} + st\boldsymbol{b}.$$

If $st = 1$, then $s = 1 = t$, so $s^{\boldsymbol{a}} = 0 = t^{\boldsymbol{a}}$ and $\boldsymbol{b} = \boldsymbol{c}$. This is the only possibility, since if $st < 1$, then

$$\boldsymbol{b} = \sum_{\boldsymbol{a}} \frac{s^{\boldsymbol{a}} + st^{\boldsymbol{a}}}{1 - st} \boldsymbol{a},$$

where

$$\sum_{\boldsymbol{a}} \frac{s^{\boldsymbol{a}} + st^{\boldsymbol{a}}}{1 - st} = \frac{1 - s + s(1 - t)}{1 - st} = 1,$$

so that $\boldsymbol{b} \in \mathrm{H}\,(A)$. □

Theorem 2. *If a convex set has a minimal spanning set, this is unique.*

Proof. Suppose

- $A \cup \{\boldsymbol{c}\}$ is a minimal spanning set for $\mathrm{H}(A \cup \{\boldsymbol{c}\})$,
- $\boldsymbol{c} \notin A$,
- $B \subseteq \mathrm{H}(A \cup \{\boldsymbol{c}\})$,
- $\boldsymbol{c} \notin B$.

We show that B does not span $\mathrm{H}(A \cup \{\boldsymbol{c}\})$. It is enough to show that, for all m, for all subsets $\{\boldsymbol{b}_i : i < m\}$ of B,

$$\boldsymbol{c} \notin \mathrm{H}(A \cup \{\boldsymbol{b}_i : i < m\}).$$

This is true by hypothesis when $m = 0$. If it is true for some k when $m = k$, then it is true when $m = k + 1$, by Lemma 7. □

Theorem 3. *The intersection of a polytope and an affine subspace of* $\mathbb{R}^n$ *is a polytope of* $\mathbb{R}^n$.

Proof. Let the polytope be K; the affine subspace, A. By Lemmas 2 and 3,

$$K \cap A \subseteq \mathrm{H}(K \cap A) \subseteq \mathrm{H}(K) \cap \mathrm{H}(A) = K \cap A,$$

so $K \cap A$ is a convex hull. To continue, since we can repeat the result as needed, we may assume A is a hyperplane. Let V be the set of vertices of K that are in A; W, not in A. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are in W, then $\mathrm{H}(\boldsymbol{a}, \boldsymbol{b}) \cap A$ contains at most one point. Let X be the set of all such points. We show

$$K \cap A = \mathrm{H}(V \cup X).$$

Say

$$W' \cup \{\boldsymbol{a}, \boldsymbol{b}\} \subseteq W,$$
$$\mathrm{H}(\boldsymbol{a}, \boldsymbol{b}) \cap A = \{\boldsymbol{c}\},$$
$$\boldsymbol{d} \in \mathrm{H}(V \cup W' \cup \{\boldsymbol{a}, \boldsymbol{b}\}).$$

In particular, $\boldsymbol{c} \in X$. By Lemma 5, there is $\boldsymbol{e}$ in $\mathrm{H}(\boldsymbol{a}, \boldsymbol{b})$ such that

$$\boldsymbol{d} \in \mathrm{H}(V \cup W' \cup \{\boldsymbol{e}\}).$$

By Lemma 6, we may now assume $\boldsymbol{e} \in \mathrm{H}(\boldsymbol{a}, \boldsymbol{c})$, and then

$$\boldsymbol{d} \in \mathrm{H}(V \cup X \cup W' \cup \{\boldsymbol{a}\}).$$

By repeating, we may assume

$$\boldsymbol{d} \in \mathrm{H}(V \cup X \cup \{\boldsymbol{a}\}).$$

By Lemma 7, if also $\boldsymbol{d} \in A$, then $\boldsymbol{d} \in \mathrm{H}(V \cup X)$. □

5 Vertices and cells

Having a finite spanning set, every polytope has a minimal spanning set. By Theorem 2, this set is unique. Its elements are the **vertices** of the polytope, in the strict sense mentioned on page 10.

A **cell** (also called a **facet**) of an n-polytope K_n in $\mathbb{R}^n$ is the intersection of the polytope with an affine hyperplane of $\mathbb{R}^n$, provided

- this intersection is an $(n-1)$-polytope,
- the original polytope K_n lies on one side of the hyperplane.

A sequence $(K_n, \dots, K_0)$ is a **flag** of K_n if each K_{i-1} is a cell of K_i. In this case, K_2 is a **face** of K_n, and K_1 is an **edge.**

For example, for every permutation π of n, we obtain a flag $(K_{n-1}, \dots, K_0)$ in $\mathbb{R}^n$ by defining

$$K_j = \mathrm{H}\left(\mathbf{e}_{\pi(0)}, \dots, \mathbf{e}_{\pi(j)}\right),$$

which is a j-simplex, when $j < n$.

Now we make an adjustment, letting also $\sigma \in \{\pm 1\}^n$, then defining

$$K_j = \mathrm{H}\left(\sigma_0 \mathbf{e}_{\pi(0)}, \dots, \sigma_j \mathbf{e}_{\pi(j)}\right),$$

again a j-simplex, when $j < n$. If we also define

$$K_n = \mathrm{H}\left((-1)^i \mathbf{e}_j \colon (i,j) \in 2 \times n\right),$$

the n-orthoplex, we have a flag $(K_n, \dots, K_0)$.

We obtain a flag of the n-cube by defining

$$K_j = \mathrm{H}\left(\sum_{i\in X} \mathbf{e}_{\pi(i)} : X \subseteq j\right),$$

which is a j-cube. Here $K_0 = \{\mathbf{0}\}$, but by symmetry we could start with any vertex. Indeed, centering the n-cube at the origin, letting $\sigma \in \{\pm 1\}^n$, we can define

$$K_j = \mathrm{H}\left(\sum_{i<n} \tau_i \mathbf{e}_{\pi(i)} : \tau \in 2^n \wedge \tau \restriction n - j = \sigma \restriction n - j\right).$$

Suppose $\boldsymbol{v}$ is a vertex of an arbitrary n-polytope K_n, where $n > 0$, and X is the set of midpoints of the edges of K_n that contain $\boldsymbol{v}$. If $\mathrm{H}(X)$ is an $(n-1)$-polytope, then it is called a **vertex figure** of K_n at $\boldsymbol{v}$.

Thus the vertex figure of

- the n-simplex is an $(n-1)$-simplex;
- the n-orthoplex is an $(n-1)$-orthoplex;
- the n-cube is an $(n-1)$-simplex;
- icosahedron is the pentagon;
- dodecahedron is the 2-simplex or triangle.

6 Regular polytopes

Every 0- and 1-polytope is **regular,** vacuously.

The **regular 2-polytopes** are the *regular polygons* in the usual sense. When $p > 2$, the **regular p-gon** is the convex hull in $\mathbb{R}^2$ of the points

$$\left(\cos\frac{2\pi k}{p}, \sin\frac{2\pi k}{p}\right),$$

where $k < p$.

The simplices, orthoplices, and cubes satisfy the following recursive definition of a **regular n-polytope** when $n > 2$:

1) each of its cells is regular,
2) the cells are isometric to each other,
3) at each vertex, there is a vertex figure,
4) each vertex figures is regular,
5) the vertex figures are isometric to each other.
6) the vertex figures of the cells are just the cells of the vertex figures,
7) there is a point—the **center**—from which all vertices are equidistant.

Some of these conditions may be redundant. In any case, the regular Platonic solids meet the conditions.

We can now denote regular polytopes (other than 0- and 1-polytopes) by **Schläfli symbols** as follows. The regular p-gon is denoted by

$$\{p\}.$$

Table 6.1: Some Schläfli symbols

The polytope called	has the Schläfli symbol
simplex	$\{3,\dots,3\}$
orthoplex	$\{3,\dots,3,4\}$
cube	$\{4,3,\dots,3\}$
icosahedron	$\{3,5\}$
dodecahedron	$\{5,3\}$

If $n \geqslant 3$, a regular n-polytope is denoted by

$$\{p_1,\dots,p_{n-1}\},$$

provided

- the cells are $\{p_1,\dots,p_{n-2}\}$,
- the vertex figures are $\{p_2,\dots,p_{n-1}\}$.

Then the faces of $\{p,\dots\}$ are p-gons. The Schläfli symbols for the polytopes in Table 2.1 are as in Table 6.1.

7 Schläfli's criterion

Letting K_n be a regular n-polytope, we write its Schläfli symbol as $\{p_{n-1}, \ldots, p_1\}$. We want to know the possibilities for the p_m. To this end, we consider successive vertex figures. If $m < n$, let K_m denote $\{p_{m-1}, \ldots, p_1\}$. Then K_m is the vertex figure of K_{m+1}. We define Δ_m recursively by

$$\begin{aligned} \Delta_1 = 1, \qquad \Delta_2 = \sin^2 \frac{\pi}{p_1}, \\ \Delta_{m+1} = \Delta_m - \Delta_{m-1} \cos^2 \frac{\pi}{p_m}. \end{aligned} \tag{7.1}$$

We shall prove $\Delta_n > 0$ by considering in K_{m+1} a triangle OMV as in Figure 7.1, where

- O is the center,
- M is the midpoint of an edge,
- V is an endpoint of that edge.

We define

$$VO = R_m, \qquad VM = \ell_m, \qquad \angle MOV = \varphi_m.$$

Here φ_m is the **central angle** of K_{m+1}.

Theorem 4 (Schläfli's Criterion)**.** *For every regular polytope* $\{p_{n-1}, \ldots, p_1\}$,

$$\Delta_n = \prod_{1 \leqslant m < n} \sin^2 \varphi_m.$$

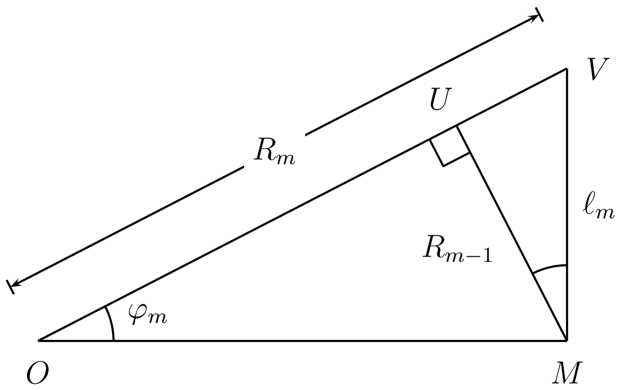

Figure 7.1: Parameters of K_{m+1} and vertex figure

Proof. Since $\Delta_1 = 1$, it is enough now to show

$$\sin^2 \varphi_{m-1} = \frac{\Delta_m}{\Delta_{m-1}}$$

when $m > 1$. By definition,

$$\frac{\Delta_{m+1}}{\Delta_m} = 1 - \frac{\Delta_{m-1}}{\Delta_m} \cos^2 \frac{\pi}{p_m},$$

so it is enough to show

$$\sin^2 \varphi_m = 1 - \frac{\cos^2(\pi/p_m)}{\sin^2 \varphi_{m-1}}. \qquad (7.2)$$

From OMV as in Figure 7.1,

$$\ell_m = R_m \sin \varphi_m.$$

We are going to use this in the form

$$\ell_{m-1} = R_{m-1} \sin \varphi_{m-1}.$$

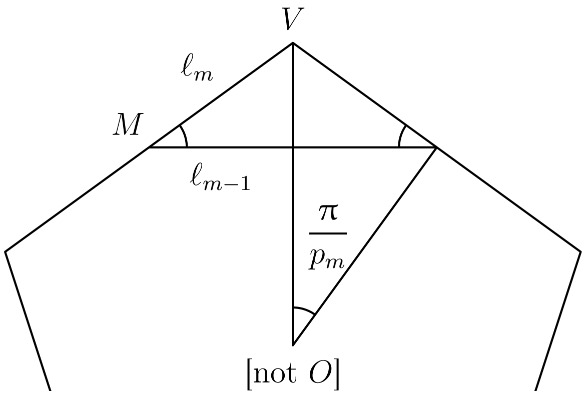

Figure 7.2: A face of K_{m+1}

We shall eliminate R_{m-1} by noting that K_{m+1} has a vertex figure with vertex M and center U on OV, to which then UM is orthogonal as in Figure 7.1. Thus

$$R_{m-1} = \ell_m \cos\varphi_m$$

and therefore

$$\ell_{m-1} = \ell_m \sin\varphi_{m-1} \cos\varphi_m.$$

There are two edges of K_{m+1} sharing V as an endpoint that are sides of a p_m-gon, as in Figure 7.2; moreover, the midpoints of these edges are the endpoints of an edge of K_m; this yields

$$\ell_{m-1} = \ell_m \cos\frac{\pi}{p_m}.$$

Therefore

$$\sin\varphi_{m-1}\cos\varphi_m = \cos\frac{\pi}{p_m}.$$

This yields (7.2) by the Pythagorean Identity. □

8 The regular polyhedra

We compute from (7.1) in particular

$$\Delta_3 = \sin^2 \frac{\pi}{p_1} - \cos^2 \frac{\pi}{p_2},$$

which by Theorem 4 yields the equivalent conditions

$$\begin{aligned} \cos \frac{\pi}{p_2} &< \sin \frac{\pi}{p_1}, \\ \sin \left(\frac{\pi}{2} - \frac{\pi}{p_2} \right) &< \sin \frac{\pi}{p_1}, \\ \frac{1}{2} &< \frac{1}{p_1} + \frac{1}{p_2}. \end{aligned}$$

Hence the Schläfli symbol of a regular polyhedron must be one of

$$\{3,3\}, \qquad \{3,4\}, \qquad \{4,3\}, \qquad \{3,5\}, \qquad \{5,3\}.$$

Moreover, such polyhedra do exist. Indeed, the first three are the 3-simplex (that is, the tetrahedron), the 3-orthoplex (the octahedron), and the 3-cube (the cube). The last two are the **icosahedron** and the **dodecahedron,** which can be defined in terms of the *golden ratio.*

8.1 Golden ratio

The **golden ratio,** denoted by φ, is determined by two conditions,

$$\varphi > 0, \qquad \varphi = \frac{1}{\varphi - 1},$$

so that also

$$\varphi^2 = \varphi + 1 \tag{8.1}$$

and hence

$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

In general, from (8.1),

$$\varphi^{n+1} = F_{n+1}\varphi + F_n,$$

where

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n.$$

The F_n are the Fibonacci numbers. Some particular powers of φ are in Table 8.1.

8.2 Icosahedron

A typical vertex of the regular icosahedron is

$$(\varphi, 1, 0).$$

The entries can be evenly permuted, and the sign of any entry can change; that makes the 12 vertices shown in Figure 8.1. We can write the vertices generally as

Table 8.1: Powers of the golden ratio

n	φ^n	φ^{-n}
0	1	1
1	φ	$\varphi - 1$
2	$\varphi + 1$	$-\varphi + 2$
3	$2\varphi + 1$	$2\varphi - 3$
4	$3\varphi + 2$	$-3\varphi + 5$

$$\sigma_0 \varphi \mathbf{e}_i + \sigma_1 \mathbf{e}_{i+1} \colon \sigma \in \{\pm 1\}^2 \wedge i \in \mathbb{Z}/(3).$$

The triangular faces fall into two families. In one family, the triangles have vertices of the form

$$(\varphi, 1, 0), \qquad (0, \varphi, 1), \qquad (0, \varphi, -1);$$

such triangles are equilateral, since

$$\varphi^2 + (\varphi - 1)^2 + 1 = 2(\varphi^2 - \varphi + 1) = 4 = 2^2.$$

In the other family, the triangles have vertices of the form

$$(\varphi, 1, 0), \qquad (0, \varphi, 1), \qquad (1, 0, \varphi),$$

so they are equilateral by symmetry. They also share sides with the first family, so all of the faces of the icosahedron are isometric.

We may also write $(\sigma_0 \mathbf{e}_i, \sigma_1 \mathbf{e}_{i+1}, \sigma_2 \mathbf{e}_{i+2})$ as $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and then $(\varphi \boldsymbol{a}, \varphi \boldsymbol{b}, \varphi \boldsymbol{c})$ as $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$. Then the two families of

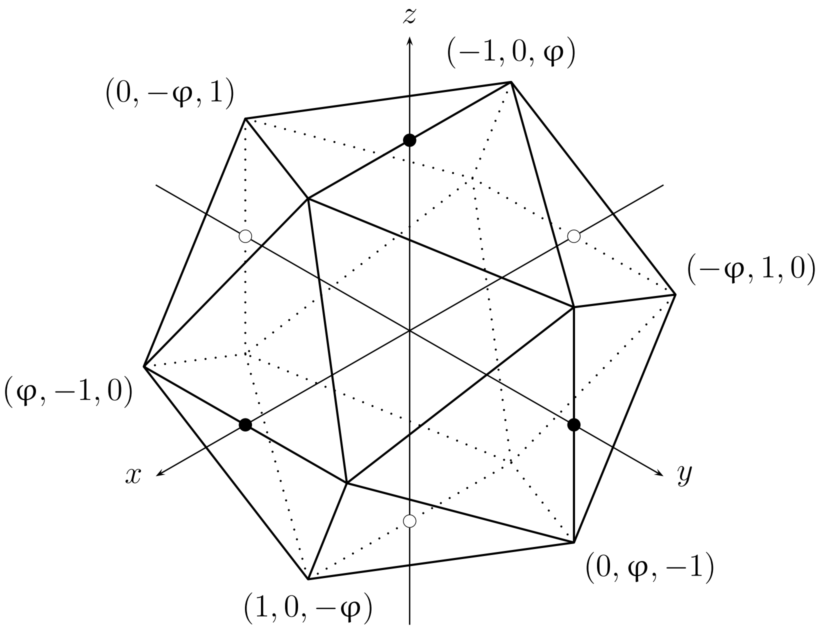

Figure 8.1: Icosahedron

faces have vertices

$$\boldsymbol{A}+\boldsymbol{b}, \qquad \boldsymbol{B}+\boldsymbol{c}, \qquad \boldsymbol{B}-\boldsymbol{c}$$

and

$$\boldsymbol{A}+\boldsymbol{b}, \qquad \boldsymbol{B}+\boldsymbol{c}, \qquad \boldsymbol{C}+\boldsymbol{a},$$

and the 12 vertices fit the icosahedron as in Figure 8.2. We still have to confirm that the vertex figures are regular pentagons. Perhaps this is intuitively clear, but for a proof, we need only check that the points

$$(\varphi,1,0), \quad (0,\varphi,-1), \quad (-\varphi,1,0), \quad (-1,0,\varphi), \quad (1,0,\varphi)$$

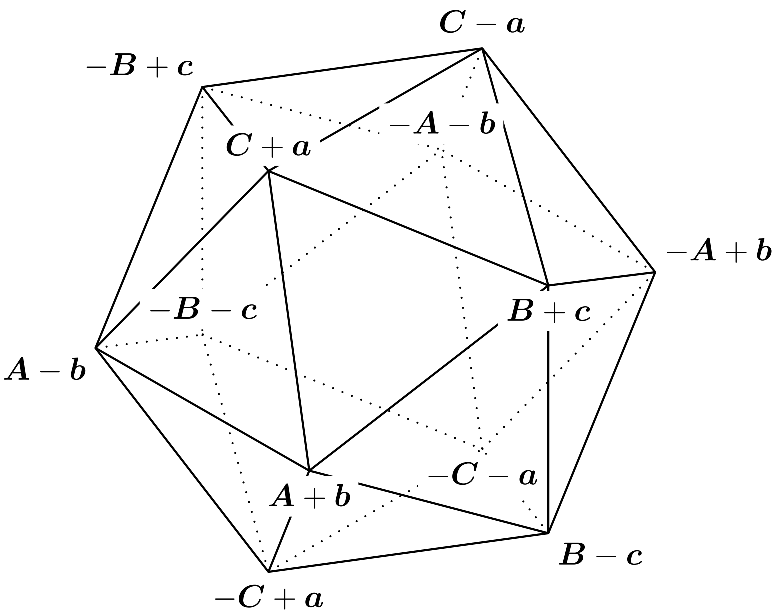

Figure 8.2: Icosahedron

are the vertices of a regular pentagon. We already know that each is the same distance from the next. For them to be coplanar, it is enough that, in $\mathbb{R}^2$, the points

$$(\varphi, -1), \qquad (1, 0), \qquad (0, \varphi)$$

be collinear; and they are, since $\varphi - 1 = \varphi^{-1}$. Finally, the first of the five points is a distance 2φ from both the third and the fourth, since

$$(\varphi + 1)^2 + 1 + \varphi^2 = 2(\varphi^2 + \varphi + 1) = 4\varphi^2;$$

and the second is the same distance from the fourth.

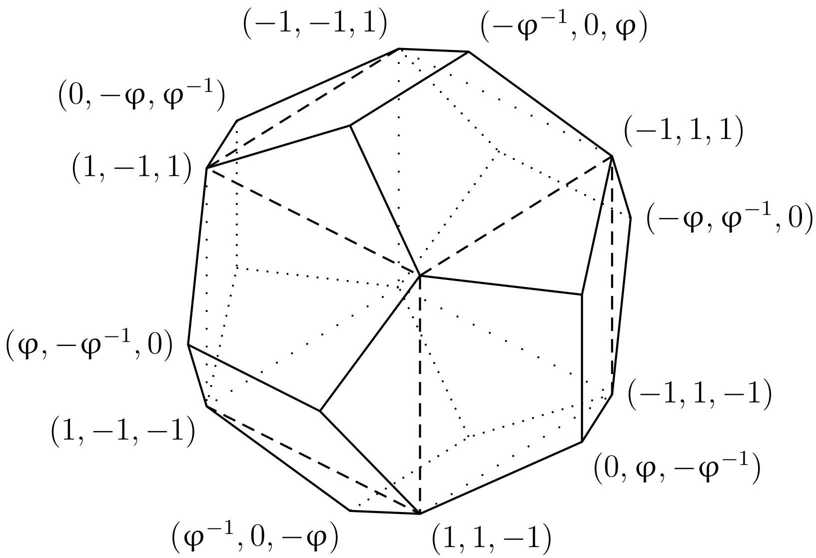

Figure 8.3: Dodecahedron

8.3 Dodecahedron

The vertices of the dodecahedron can be the centers of the faces of the icosahedron. These centers are of the forms

$$\frac{1}{3}(\boldsymbol{A} + 2\boldsymbol{B} + \boldsymbol{b}), \qquad \frac{1}{3}(\boldsymbol{A} + \boldsymbol{a} + \boldsymbol{B} + \boldsymbol{b} + \boldsymbol{C} + \boldsymbol{c}),$$

which are

$$\frac{\varphi^2}{3}(\varphi^{-1}\boldsymbol{a} + \varphi\boldsymbol{b}), \qquad \frac{\varphi^2}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}),$$

Scaling, we obtain the regular dodecahedron that has for vertices the 20 points shown in Figure 8.3, namely

$$\begin{aligned}&\sigma_0\mathbf{e}_0+\sigma_1\mathbf{e}_1+\sigma_2\mathbf{e}_2\colon \sigma\in\{\pm1\}^3,\\&\sigma_0\varphi\mathbf{e}_i+\sigma_1\varphi^{-1}\mathbf{e}_{i+1}\colon \sigma\in\{\pm1\}^2\wedge i\in\mathbb{Z}/(3).\end{aligned}$$

9 The regular polychora

We compute next

$$\begin{aligned}\Delta_4 &= \sin^2\frac{\pi}{p_1} - \cos^2\frac{\pi}{p_2} - \sin^2\frac{\pi}{p_1}\cos^2\frac{\pi}{p_3}\\ &= \sin^2\frac{\pi}{p_1}\sin^2\frac{\pi}{p_3} - \cos^2\frac{\pi}{p_2},\end{aligned}$$

which yields the condition

$$\sin\frac{\pi}{p_3}\sin\frac{\pi}{p_1} > \cos\frac{\pi}{p_2}. \tag{9.1}$$

We can solve this directly, using the values in Table 9.1.

- When $p_3 = p_1 = 3$, then $\sin(\pi/p_3)\sin(\pi/p_1) = 3/4$, so $p_2 \in \{3, 4\}$.
- When $p_3 = p_1 = 4$, then $\sin(\pi/p_3)\sin(\pi/p_1) = 1/2$, so p_2 can be nothing.
- When $\{p_1, p_3\} = \{3, 6\}$, then

  $$\sin\frac{\pi}{p_3}\sin\frac{\pi}{p_1} = \frac{\sqrt{3}}{4} < \frac{1}{2},$$

  so p_2 can be nothing.
- When $\{p_1, p_3\} = \{3, 5\}$ or $\{p_1, p_3\} = \{3, 4\}$, then

  $$0.62 > \frac{\sqrt{6}}{4} \geqslant \sin\frac{\pi}{p_3}\sin\frac{\pi}{p_1} \geqslant \frac{\sqrt{6(5-\sqrt{5})}}{8} > 0.5,$$

  so $p_2 = 3$.

Table 9.1: Sines and cosines

p	$\sin\frac{\pi}{p}$		$\cos\frac{\pi}{p}$		$\frac{2\pi}{p}$
3	$\frac{\sqrt{3}}{2}$	≈ 0.87	$\frac{1}{2}$	$= 0.50$	$120°$
4	$\frac{\sqrt{2}}{2}$	≈ 0.71	$\frac{\sqrt{2}}{2}$	≈ 0.71	$90°$
5	$\frac{\sqrt{2(5-\sqrt{5})}}{4}$	≈ 0.59	$\frac{1+\sqrt{5}}{4}$	≈ 0.81	$72°$
6	$\frac{1}{2}$	$= 0.50$	$\frac{\sqrt{3}}{2}$	≈ 0.87	$60°$

We can also understand (9.1) in terms of the *dihedral* angle of $\{p_3, p_2\}$. Let A be a vertex of this polyhedron, and let the corresponding vertex figure $\{p_2\}$ be $BCD\ldots$. Let E be the foot of the perpendicular dropped from B to CA (after CA is extended if necessary). Then DE is also perpendicular to CA, and the dihedral angle of $\{p_3, p_2\}$ is BED. Let $\theta(p_3, p_2)$ be half this angle. Then

$$\sin\theta(p_3, p_2) = \frac{BD}{2BE}.$$

Let us consider the half-edge of $\{p_3, p_2\}$ as a unit. Since the polygon $BAC\ldots$ is $\{p_3\}$, we have

$$BE = \sin\frac{2\pi}{p_3} = 2\sin\frac{\pi}{p_3}\cos\frac{\pi}{p_3}.$$

Since also the polygon $DAC\ldots$ is $\{p_3\}$, we have

$$DC = BC = 2\cos\frac{\pi}{p_3},$$

and therefore, since the polygon $BCD\ldots$ is $\{p_2\}$,

$$BD = 2BC\cos\frac{\pi}{p_2} = 4\cos\frac{\pi}{p_3}\cos\frac{\pi}{p_2}.$$

Putting this all together gives

$$\sin\theta(p_3,p_2) = \frac{\cos(\pi/p_2)}{\sin(\pi/p_3)}.$$

Schläfli's criterion is now

$$\sin(\pi/p_1) > \sin\theta(p_3,p_2),$$

or

$$\frac{2\pi}{p_1} > 2\theta(p_3,p_2);$$

that is, it must be possible to arrange a number p_1 of polyhedra $\{p_3,p_2\}$ so as to have a common edge.

To check when this is possible, we need the values in Table 9.2. Therefore the regular polychora are among

$$\{3,3,3\}\quad \{3,3,4\},\quad \{4,3,3\},\quad \{3,3,5\},\quad \{5,3,3\},\quad \{3,4,3\};$$

but existence must be established. The first three of the possibilities are the 4-simplex, the 4-orthoplex, and the 4-cube.

Table 9.2: Dihedral angles

$\{p,q\}$	$\sin\theta(p,q)$	$2\theta(p,q)$
$\{3,3\}$	$\frac{\sqrt{3}}{3}$	$\approx 70.53^\circ$
$\{3,4\}$	$\frac{\sqrt{6}}{3}$	$\approx 109.47^\circ$
$\{4,3\}$	$\frac{\sqrt{2}}{2}$	90°
$\{3,5\}$	$\frac{\sqrt{3}+\sqrt{15}}{6}$	$\approx 138^\circ$
$\{5,3\}$	$\frac{\sqrt{10(5+\sqrt{5})}}{10}$	$\approx 116^\circ$

10 The regular 24-cell

We are going to show that $\{3,4,3\}$ exists as the convex hull in $\mathbb{R}^4$ of 24 points, namely those with just two entries, each of these being ± 1. We can write those vertices in several ways:

- as
$$\sigma_0 \mathbf{e}_i + \sigma_1 \mathbf{e}_j \colon \sigma \in \{\pm 1\}^2 \wedge i < j < 4;$$
- as
$$(1, 1, 0, 0),$$
where we allow the entries to be permuted and allow any entry to change sign;
- as
$$\boldsymbol{a} + \boldsymbol{b},$$
where each of $\boldsymbol{a}$ and $\boldsymbol{b}$ is a distinct element, or its negative, of $\{\mathbf{e}_i \colon i < 4\}$.

We proceed as follows.

1. The polychoron spanned by the given points is bounded by each of the hyperplanes defined respectively by the 8 equations

$$\sigma x_i = 1 \colon \sigma \in \{\pm 1\} \wedge i < 4 \tag{10.1}$$

and the 16 equations

$$\sigma_0 x_0 + \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3 = 2 \colon \sigma \in \{\pm 1\}^4. \tag{10.2}$$

a) The hyperplane defined by

$$x_0 = 1$$

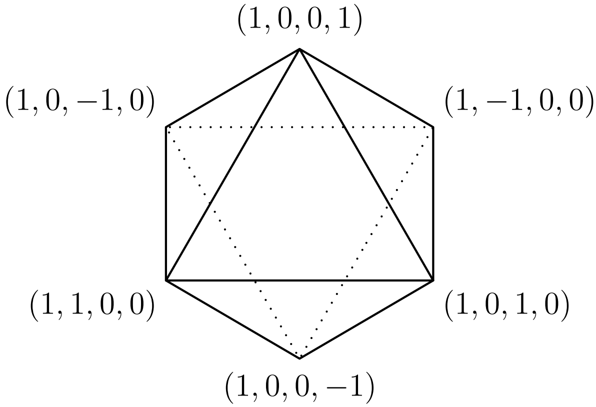

Figure 10.1: Octahedron in hyperplane $x_0 = 1$

intersects the polychoron in the octahedron that is the convex hull of the 6 points shown in Figure 10.1. Using any of the equations (10.1), we get the octagon in whose vertices

- entry i is always σ (which is ± 1);
- just one other entry is nonzero, and this is ± 1.

We can depict that octagon as in Figure 10.2, where

$$\boldsymbol{a} = \sigma \mathbf{e}_i, \qquad \{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\} = \big\{\mathbf{e}_j : j \in 4 \smallsetminus \{i\}\big\}.$$

b) The hyperplane defined by

$$x_0 + x_1 + x_2 + x_3 = 2$$

intersects the polychoron in the octahedron that is the convex hull of the 6 points shown in Figure 10.3. Using any of the equations (10.2), we get the octagon, each of whose vertices is obtained from $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ by making

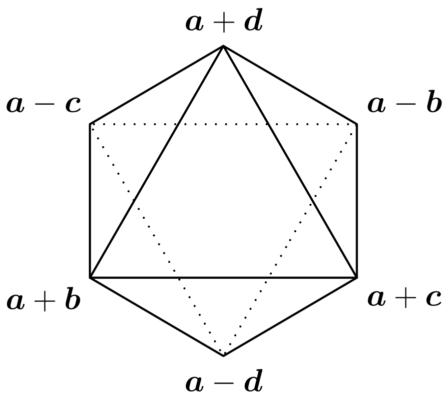

Figure 10.2: Octahedron in hyperplane $\sigma x_i = 1$

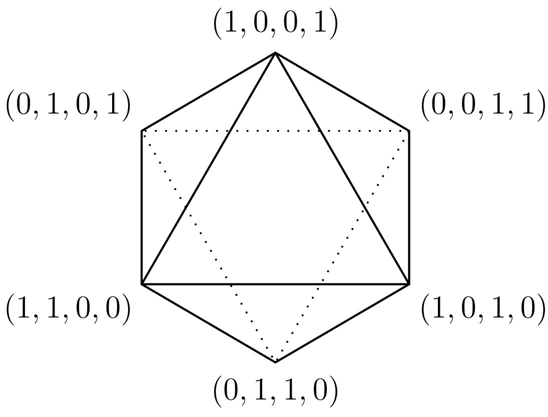

Figure 10.3: Octahedron in hyperplane $x_0 + x_1 + x_2 + x_3 = 2$

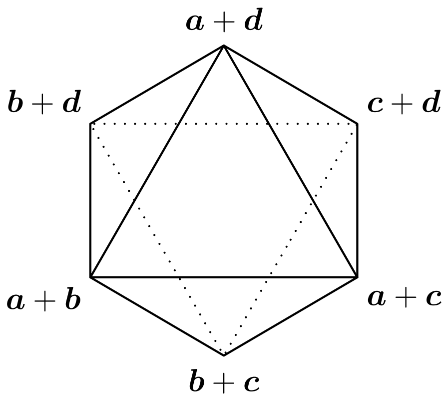

Figure 10.4: Octahedron in hyperplane $\sum_{i<4}\sigma_i x_i = 2$

two entries 0. We can depict that octagon as in Figure 10.4, where

$$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\} = \{\sigma_i \mathbf{e}_i \colon i < 4\}. \qquad (10.3)$$

We have now identified 24 cells of our polychoron. Each of them is an octagon of edge length $\sqrt{2}$.

2. Every vertex of the polychoron belongs to 6 of the hyperplanes so far mentioned, so it is a common vertex of 6 of the octahedra. For example, $(1, 1, 0, 0)$ belongs to each of the hyperplanes defined respectively by

$$x_i = 1 \colon i < 2,$$
$$x_0 + x_1 + \sigma_0 x_2 + \sigma_1 x_3 = 2 \colon \sigma \in \{\pm 1\}^2.$$

3. There are 8 vertices at distance $\sqrt{2}$ from a given vertex, and they are the vertices of a cube, each of whose edges has length $\sqrt{2}$. For example, $(1, 1, 0, 0)$ is $\sqrt{2}$ away from the

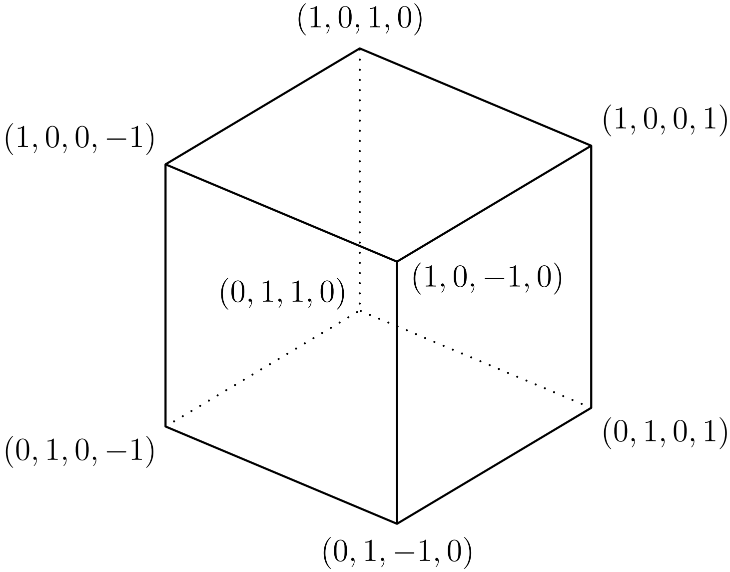

Figure 10.5: Cube with vertices $\sqrt{2}$ from $(1, 1, 0, 0)$

vertices of the cube in Figure 10.5. Note that the center of the cube is $(1/2, 1/2, 0, 0)$, not $(1, 1, 0, 0)$. Alternatively, in the notation of (10.3) used for Figure 10.4, the point $\boldsymbol{a} + \boldsymbol{b}$ is equidistant from the vertices of the cube in Figure 10.6.

4. Each vertex of the cube belongs to one of the 6 octahedra meeting at the given vertex. In fact it belongs to 3 of them, since, with that given vertex, the vertices of each face of the cube belong to one of the octahedra. In particular, the octahedra meeting at the given vertex surround it, so that it can belong to no other cell.

Thus the cells of the polychoron are just the 24 octahedra that we identified, and there is a vertex figure, namely a cube,

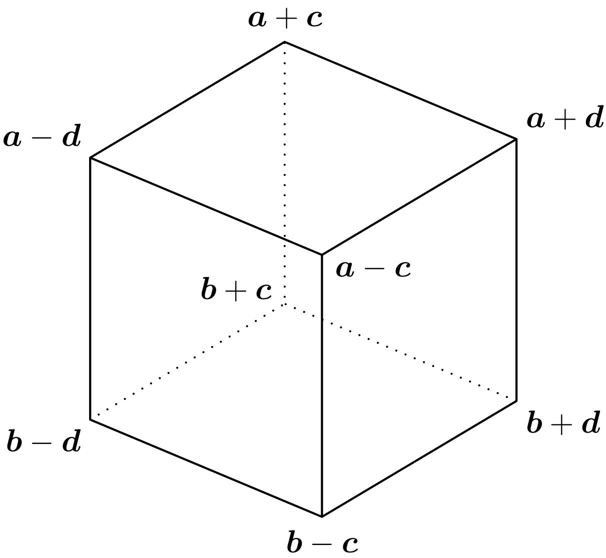

Figure 10.6: Cube with vertices $\sqrt{2}$ from $\boldsymbol{a}+\boldsymbol{b}$

at each vertex. The octahedron being $\{3,4\}$; and the cube, $\{4,3\}$; our polychoron is $\{3,4,3\}$. This is called the 24**-cell.**

11 The regular 600-cell

For $\{3,3,5\}$, we take the convex hull of

$$\begin{aligned}\sigma_0\varphi\mathbf{e}_{\tau(0)}+\sigma_1\mathbf{e}_{\tau(1)}+\sigma_2\varphi^{-1}\mathbf{e}_{\tau(2)}&\colon\sigma\in\{\pm1\}^3\wedge\tau\in\operatorname{Alt}(4),\\ \sigma 2\mathbf{e}_i&\colon\sigma\in\{\pm1\}\wedge i<4,\\ \sigma_0\mathbf{e}_0+\sigma_1\mathbf{e}_1+\sigma_2\mathbf{e}_2+\sigma_3\mathbf{e}_3&\colon\sigma\in\{\pm1\}^4.\end{aligned}$$

We compute the number of these points:

$$2^3\cdot12+2\cdot4+2^4=2^3\cdot(12+1+2)=2^3\cdot3\cdot5=120.$$

Typical examples are

$$(\varphi,1,\varphi^{-1},0),\qquad(2,0,0,0),\qquad(1,1,1,1);$$

the entries can be permuted evenly, and any entries can change sign. Alternatively, we may write the typical points as

$$\varphi\boldsymbol{a}+\boldsymbol{b}+\varphi^{-1}\boldsymbol{c},\qquad2\boldsymbol{a},\qquad\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d},$$

again using (10.3), but we must understand that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ is an *even* permutation of $(\sigma_0\mathbf{e}_0,\sigma_1\mathbf{e}_1,\sigma_2\mathbf{e}_2,\sigma_3\mathbf{e}_3)$. We may also write

- $\varphi\boldsymbol{a}$, $\varphi\boldsymbol{b}$, $\varphi\boldsymbol{c}$, and $\varphi\boldsymbol{d}$ as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$;
- $\varphi^{-1}\boldsymbol{a}$, $\varphi^{-1}\boldsymbol{b}$, $\varphi^{-1}\boldsymbol{c}$, and $\varphi^{-1}\boldsymbol{d}$ as $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{c}'$, and $\boldsymbol{d}'$.

Thus $\varphi\boldsymbol{a}+\boldsymbol{b}+\varphi^{-1}\boldsymbol{c}$ is

$$\boldsymbol{A}+\boldsymbol{b}+\boldsymbol{c}'.$$

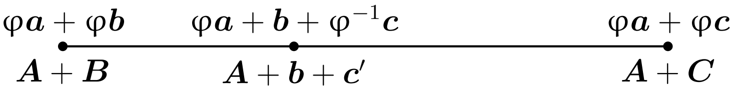

Figure 11.1: Division in the golden ratio

We first look at the 96 points of this form. Each one divides the segment having endpoints $\boldsymbol{A}+\boldsymbol{B}$ and $\boldsymbol{A}+\boldsymbol{C}$ in the golden ratio, as depicted in Figure 11.1, since

$$\varphi = 1 + \varphi^{-1}.$$

In particular, the three points

$$(\varphi, \varphi, 0, 0), \qquad (\varphi, 1, \varphi^{-1}, 0), \qquad (\varphi, 0, \varphi, 0)$$

demarcate a segment so divided. The divided segments are the edges of our 24-cell, scaled by φ.

When we divide the sides of an octahedron in the golden ratio, in the right sense for each edge, we obtain an icosahedron, as in Figure 11.2. When the octahedron is scaled by φ from that of Figures 10.1 and 10.2, we obtain the points shown in Figures 11.3 and 11.5; in Figures 10.3 and 10.4, the points shown in Figures 11.5 and 11.6.

In the icosahedron of Figure 11.3, we compute the distance between

$$(\varphi, 0, -1, \varphi^{-1}), \qquad (\varphi, -\varphi^{-1}, 0, 1)$$

thus:

$$\begin{aligned}\sqrt{\varphi^{-2} + 1 + (1 - \varphi^{-1})^2} &= \sqrt{(\varphi - 1)^2 + 1 + (2 - \varphi)^2} \\ &= \sqrt{2\varphi^2 - 6\varphi + 6} = \sqrt{8 - 4\varphi} = 2\varphi^{-1}.\end{aligned}$$

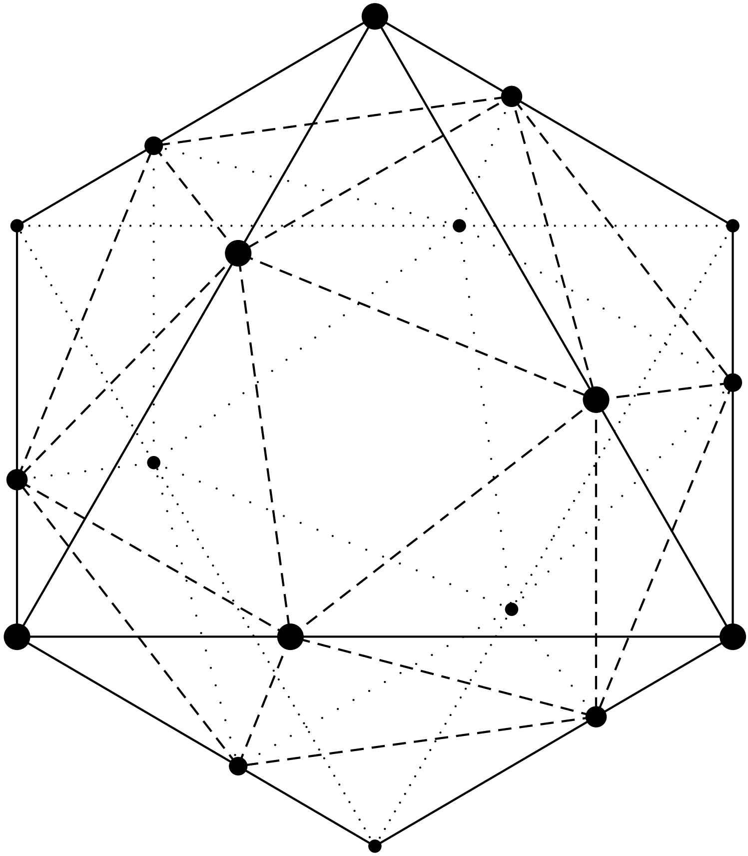

Figure 11.2: Icosahedron from octahedron

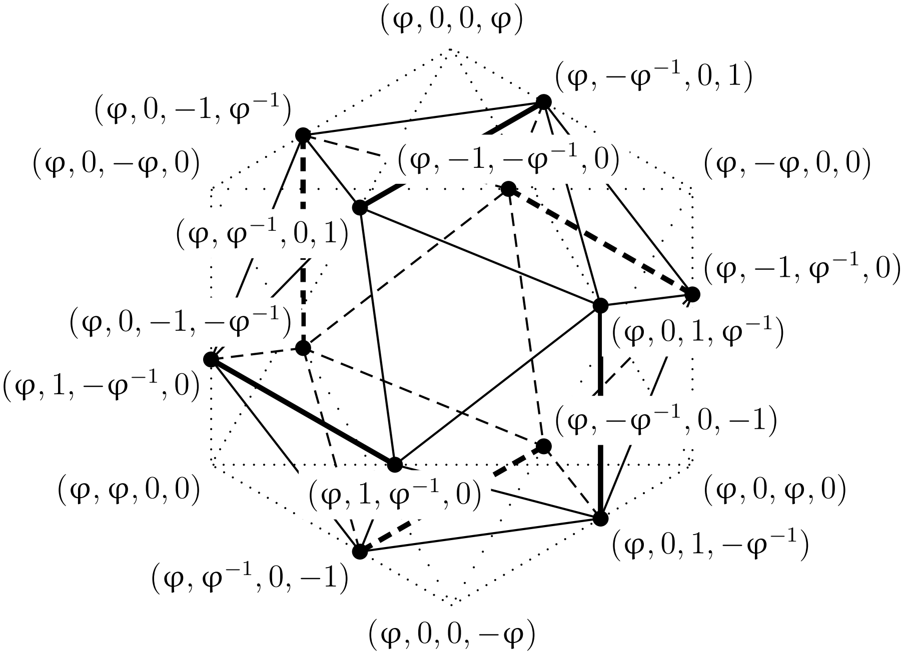

Figure 11.3: Icosahedron from Figure 10.1

This is also the distance of any vertex of the icosahedron from $(2, 0, 0, 0)$, that distance being

$$\sqrt{(2-\varphi)^2 + 1 + \varphi^{-2}}.$$

Likewise, $(1, 1, 1, 1)$ is at this distance from each of the vertices of the icosahedron in Figure 11.5.

Together, the points $\boldsymbol{A} + \boldsymbol{b} + \boldsymbol{c}'$ are the vertices of 24 icosahedra, one for each cell of the 24-cell. There is an icosahedron for each of the 8 points $2\boldsymbol{a}$, and for each of the 16 points $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}$; and each icosahedron yields 20 tetrahedra. Thus we have 480 cells so far.

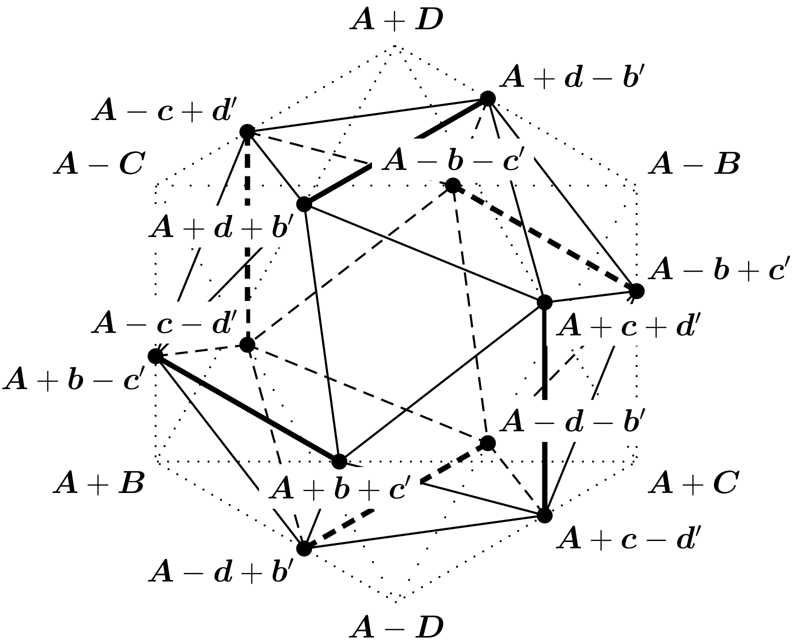

Figure 11.4: Icosahedron from Figure 10.2

Moreover, in dividing the sides of the 24-cell, we are effectively distorting each vertex figure, which was a cube originally, into five tetrahedra, as in Figure 11.7. This gives us 120 new tetrahedral cells, for a total of 600. The vertex figure is the icosahedron, $\{3, 5\}$. Our polychoron, the 600-**cell,** is thus $\{3, 3, 5\}$.

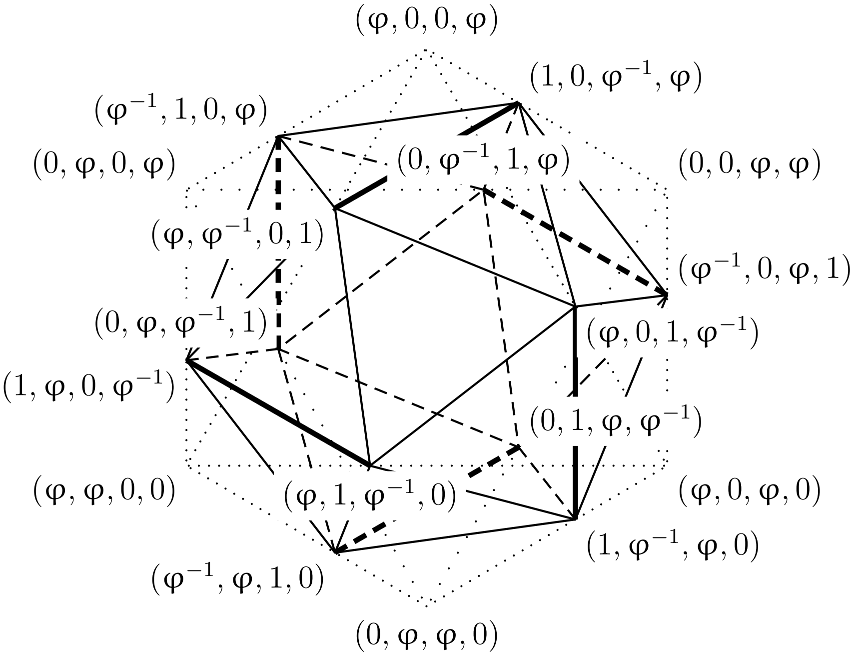

Figure 11.5: Icosahedron from Figure 10.3

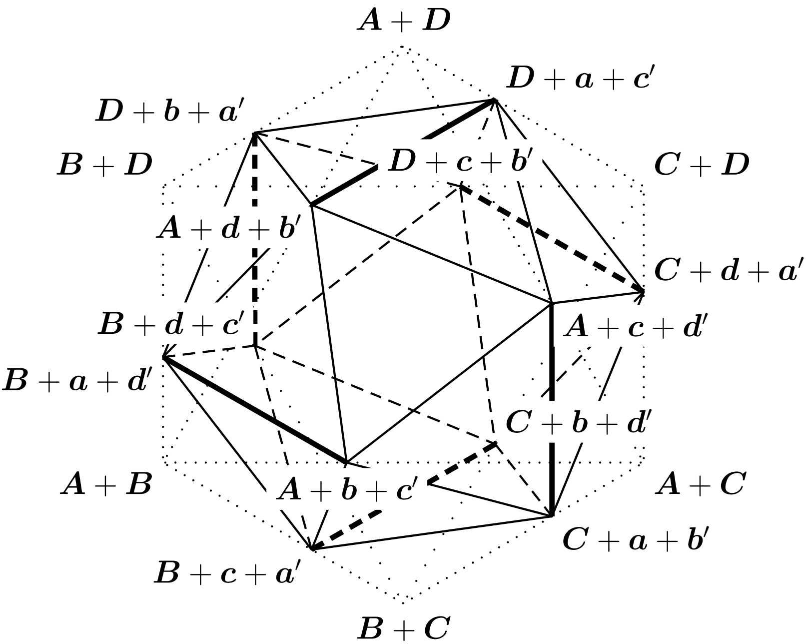

Figure 11.6: Icosahedron from Figure 10.4

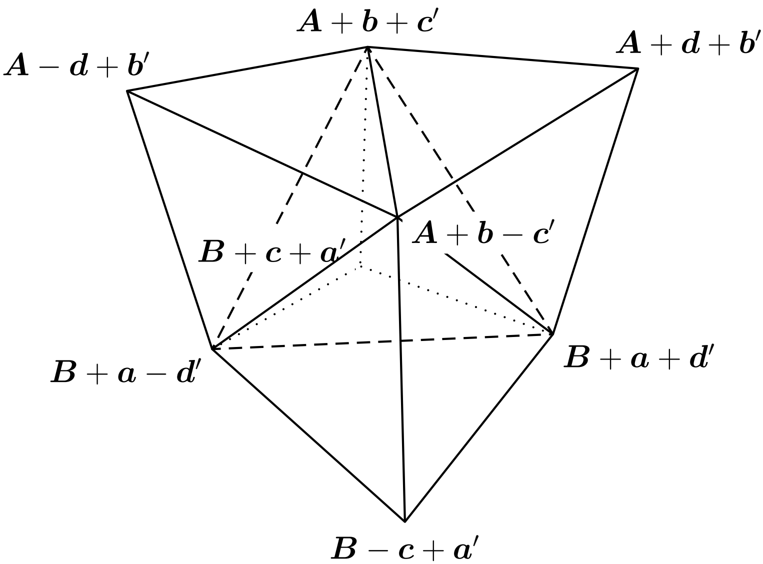

Figure 11.7: Distorted vertex figure for $\boldsymbol{A}+\boldsymbol{B}$

12 The regular 120-cell

We obtain $\{5,3,3\}$ by taking, as vertices, the centers of the cells of the 600-cell. We find those centers now.

1. With bases in the icosahedra of Figure 11.4,

a) with 8 choices for $\boldsymbol{a}$, then 6 for $\boldsymbol{b}$, then 2 for $\boldsymbol{d}$, there are 96 cells with vertices of the form

$$\boldsymbol{A}+\boldsymbol{b}+\boldsymbol{c}', \qquad \boldsymbol{A}+\boldsymbol{b}-\boldsymbol{c}', \qquad \boldsymbol{A}+\boldsymbol{d}+\boldsymbol{b}', \qquad 2\boldsymbol{a},$$

having center, scaled by $4\varphi^{-2}$,

$$\varphi^2\boldsymbol{a}+\boldsymbol{b}+\varphi^{-2}\boldsymbol{d};$$

b) with 8 choices for $\boldsymbol{a}$, then 8 for $\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\}$, there are 64 cells with vertices of the form

$$\boldsymbol{A}+\boldsymbol{b}+\boldsymbol{c}', \qquad \boldsymbol{A}+\boldsymbol{c}+\boldsymbol{d}', \qquad \boldsymbol{A}+\boldsymbol{d}+\boldsymbol{b}', \qquad 2\boldsymbol{a},$$

having center, scaled by $4\varphi^{-2}$,

$$\varphi^2\boldsymbol{a}+\varphi^{-1}\boldsymbol{b}+\varphi^{-1}\boldsymbol{c}+\varphi^{-1}\boldsymbol{d}.$$

2. With bases in the icosahedra of Figure 11.6,

a) with 8 choices for $\boldsymbol{a}$, then 6 for $\boldsymbol{b}$, then 4 for $\{\boldsymbol{c},\boldsymbol{d}\}$, there are 192 cells with vertices of the form

$$\boldsymbol{A}+\boldsymbol{b}+\boldsymbol{c}', \quad \boldsymbol{B}+\boldsymbol{a}+\boldsymbol{d}', \quad \boldsymbol{A}+\boldsymbol{d}+\boldsymbol{b}', \quad \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d},$$

having center, scaled by $4\varphi^{-2}$,

$$2\boldsymbol{a}+\varphi\boldsymbol{b}+\varphi^{-1}\boldsymbol{c}+\boldsymbol{d};$$

b) with 8 choices for $\boldsymbol{a}$, then 8 choices for $\{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$, there are 64 cells with vertices of the form

$$\boldsymbol{A} + \boldsymbol{b} + \boldsymbol{c}', \quad \boldsymbol{A} + \boldsymbol{c} + \boldsymbol{d}', \quad \boldsymbol{A} + \boldsymbol{d} + \boldsymbol{b}', \quad \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d},$$

having center, scaled by $4\varphi^{-2}$,

$$\sqrt{5}\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d};$$

c) with 8 choices for $\boldsymbol{d}$, then 8 choices for $\{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$, there are 64 tetrahedra with vertices of the form

$$\boldsymbol{A} + \boldsymbol{b} + \boldsymbol{c}', \quad \boldsymbol{B} + \boldsymbol{c} + \boldsymbol{a}', \quad \boldsymbol{C} + \boldsymbol{a} + \boldsymbol{b}', \quad \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d},$$

having center, scaled by $4\varphi^{-2}$,

$$\varphi\boldsymbol{a} + \varphi\boldsymbol{b} + \varphi\boldsymbol{c} + \varphi^{-2}\boldsymbol{d}.$$

3. In Figure 11.7,

a) with 24 choices for $\{\boldsymbol{a}, \boldsymbol{b}\}$, there are 24 tetrahedra with vertices of the form

$$\boldsymbol{A} + \boldsymbol{b} + \boldsymbol{c}', \quad \boldsymbol{A} + \boldsymbol{b} - \boldsymbol{c}', \quad \boldsymbol{B} + \boldsymbol{a} + \boldsymbol{d}', \quad \boldsymbol{B} + \boldsymbol{a} - \boldsymbol{d}',$$

having center, scaled by $4\varphi^{-2}$,

$$2\boldsymbol{a} + 2\boldsymbol{b};$$

b) with 8 choices for $\boldsymbol{a}$, then 6 for $\boldsymbol{b}$, then 2 for $\boldsymbol{d}$, there are 96 tetrahedra with vertices of the form

$$\boldsymbol{A} + \boldsymbol{b} + \boldsymbol{c}', \quad \boldsymbol{A} + \boldsymbol{b} - \boldsymbol{c}', \quad \boldsymbol{A} + \boldsymbol{d} + \boldsymbol{b}', \quad \boldsymbol{B} + \boldsymbol{a} + \boldsymbol{d}',$$

having center, scaled by $4\varphi^{-2}$,

$$\sqrt{5}\boldsymbol{a} + \varphi\boldsymbol{b} + \varphi^{-1}\boldsymbol{d}.$$

The number of centers is thus

$$96 + 64 + 192 + 64 + 64 + 24 + 96,$$

or 600.

13 Higher dimensions

The list of regular 4-polytopes restricts the possibilities for regular 5-polytopes to

$$\begin{gathered}\{3,3,3,3\},\\ \{4,3,3,3\},\quad \{3,3,3,4\},\\ \{5,3,3,3\},\quad \{3,3,3,5\},\\ \{4,3,3,4\},\\ \{5,3,3,4\},\quad \{4,3,3,5\},\\ \{5,3,3,5\},\\ \{3,3,4,3\},\quad \{3,4,3,3\}.\end{gathered}$$

Schläfli's criterion is

$$\cos^2\frac{\pi}{p_m} < \frac{\Delta_m}{\Delta_{m-1}},$$

so that, in particular,

$$\cos^2\frac{\pi}{p_4} < \frac{\Delta_4}{\Delta_3} = \frac{\sin^2(\pi/p_1)\sin^2(\pi/p_3) - \cos^2(\pi/p_2)}{\sin^2(\pi/p_1) - \cos^2(\pi/p_2)}.$$

The possibilities are in Table 13.1. The only possibilities are the ones we know,

$$\{3,3,3,3\},\qquad \{4,3,3,3\},\qquad \{3,3,3,4\}.$$

Table 13.1: Schläfli's criterion for regular 5-polytopes

(p_3, p_2, p_1)	$\dfrac{\sin^2(\pi/p_1)\sin^2(\pi/p_3) - \cos^2(\pi/p_2)}{\sin^2(\pi/p_1) - \cos^2(\pi/p_2)}$	
$(3, 3, 3)$	$\dfrac{(3/4)(3/4) - 1/4}{3/4 - 1/4}$	$5/8$
$(3, 3, 4)$	$\dfrac{(1/2)(3/4) - 1/4}{1/2 - 1/4}$	$1/2$
$(3, 3, 5)$	$\dfrac{((5-\sqrt{5})/8)(3/4) - 1/4}{(5-\sqrt{5})/8 - 3/4}$	< 0
$(3, 4, 3)$	$\dfrac{(3/4)(3/4) - 1/2}{3/4 - 1/2}$	$1/4$

The higher-dimensional regular polytopes are therefore only the ones we know.

Meanwhile, for a neater expression of Schäfli's criterion for regular 5-polytopes, we compute

$$\begin{aligned}
\Delta_5 &= \sin^2\frac{\pi}{p_1}\sin^2\frac{\pi}{p_3} - \cos^2\frac{\pi}{p_2} - \left(\sin^2\frac{\pi}{p_1} - \cos^2\frac{\pi}{p_2}\right)\cos^2\frac{\pi}{p_4} \\
&= \sin^2\frac{\pi}{p_1}\left(\sin^2\frac{\pi}{p_3} - \cos^2\frac{\pi}{p_4}\right) - \cos^2\frac{\pi}{p_2}\sin^2\frac{\pi}{p_4} \\
&= \sin^2\frac{\pi}{p_1}\left(\sin^2\frac{\pi}{p_4} - \cos^2\frac{\pi}{p_3}\right) - \cos^2\frac{\pi}{p_2}\sin^2\frac{\pi}{p_4} \\
&= \sin^2\frac{\pi}{p_1}\sin^2\frac{\pi}{p_4} - \sin^2\frac{\pi}{p_1}\cos^2\frac{\pi}{p_3} - \cos^2\frac{\pi}{p_2}\sin^2\frac{\pi}{p_4}
\end{aligned}$$

so that the criterion is

$$\frac{\cos^2(\pi/p_3)}{\sin^2(\pi/p_4)} + \frac{\cos^2(\pi/p_2)}{\sin^2(\pi/p_1)} < 1.$$

14 Sources

My main source has been Coxeter:

- Chapter 22, "Four-dimensional geometry," of *Introduction to Geometry* [5, pp. 397–414];
- Chapter VII, "Ordinary Polytopes in Higher Space," of *Regular Polytopes.*

Originally, of the latter, I had the third edition [4]. According to the brief unsigned review in *Mathematical Reviews,* compared to the second edition, the third edition

> embodies more than 20 small improvements, the bibliography has been brought up to date, and the author has added a number of comments in the new preface.

As for the second edition [3], according to the review by G. de B. Robinson,

> The author has inserted a new definition of a polytope at the end of the last chapter [presumably Chapter XIV, "Star Polytopes,"] but has omitted any reference to Barlow's work though he is quoted twice in the next. This reviewer would like to have seen some reference made to Weyl's beautiful book on symmetry . . .

Now I have only the first edition, from which I take the following:

> Practically all the ideas in this chapter . . . are due to Schläfli, who discovered them before 1853—a time when

> Cayley, Grassmann, and Möbius were the only other people who had ever conceived the possibility of geometry in more than three dimensions.
>
> . . . Ludwig Schläfli was born in Grasswyl, Switzerland, in 1814. In his youth he studied science and theology at Berne, but received no adequate instruction in mathematics . . .

Coxeter goes on to suggest a belief in phrenology or physiognomy, saying of Schläfli [2, pp. 141–2],

> His portrait shows the high forehead and keen features of a great thinker.

Other sources that I have looked at are:

- Stillwell, "The story of the 120-cell," for history and the associated group theory [7];
- Comes, "Regular polytopes," as a classification using angles (and a fair amount of geometric intuition, it would seem) [1];
- Wikipedia.

Bibliography

[1] Jonathan Comes. Regular polytopes. *The Montana Mathematics Enthusiast*, 1(2):30–37, Oct. 2004.

[2] H. S. M. Coxeter. *Regular Polytopes*. Methuen & Co., Ltd., London; Pitman Publishing Corporation, New York, 1948; 1949.

[3] H. S. M. Coxeter. *Regular polytopes.* Second edition. The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1963.

[4] H. S. M. Coxeter. *Regular polytopes.* Dover Publications, New York, third edition, 1973.

[5] H. S. M. Coxeter. *Introduction to geometry.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1969 edition.

[6] R. Tyrrell Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[7] John Stillwell. The story of the 120-cell. *Notices Amer. Math. Soc.*, 48(1):17–24, 2001.