History of mathematics

Log of a course

David Pierce

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Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
http://math.msgsu.edu.tr/~dpierce/

Matematik Bölümü
Mimar Sinan Güzel Sanatlar Üniversitesi
Bomonti, Şişli
İstanbul 34380 Turkey
http://math.msgsu.edu.tr/~dpierce/
dpierce@msgsu.edu.tr

Preface

This is about a course, given in Ankara in 2009–10, where third- and fourth-year undergraduate students read and presented original sources of mathematics. More details are in the Prolegomena, where it is explained for example that, in the Table of Contents, dates in Part I are of emails; Part II, of classes.

I moved to Istanbul in 2011, and here I have been involved in a course where first-year undergraduates read and present Book I of Euclid's *Elements.* For this course, I occasionally consult the present document, and in so doing, I may make minor adjustments, and I may add more diagrams, so that the discussion of Euclid becomes more self-contained.

I have likewise consulted, adjusted, and augmented the document while teaching an upper-level course on Galois theory.

Web pages in the domain `metu.edu.tr` are obsolete; but this should now be replaceable with `mat.msgsu.edu.tr`.

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Prolegomena

What is here

This book is a record of a course in the history of mathematics, held at Middle East Technical University, Ankara, Turkey, during the 2009/10 academic year. Officially the course was

(1) Math 303, History of Mathematical Concepts I, in the fall semester;
(2) Math 304, History of Mathematical Concepts II, in the spring.

There were about twenty students in each semester; but only four students took both semesters. The two semesters correspond to the two numbered parts of this book. According to the catalogue, the course content is thus:

> [Math 303:] Mathematics in Egypt and Mesopotamia, Ionia and Pythagoreans, paradoxes of Zeno and the heroic age. Mathematical works of Plato, Aristotle, Euclid of Alexandria, Archimedes, Apollonius and Diophantus. Mathematics in China and India. [Math 304:] Mathematics of the Renaissance, Islamic contributions. Solution of the cubic equation and consequences. Invention of logarithms. Time of Fermat and Descartes. Development of the limit concept. Newton and Leibniz. The age of Euler. Contributions of Gauss and Cauchy. Non-Euclidean geometries. The arithmetization of analysis. The rise of abstract algebra. Aspects of the twentieth century.

Most parts of this description correspond to chapter titles in the suggested textbook by Boyer [11]. But I did not use a modern textbook. My way of teaching the course was inspired by my experience at St John's College, with campuses in Annapolis, Maryland, and Santa Fe, New Mexico, USA. As a student at St John's, I learned mathematics by reading, presenting, and discussing the works of Euclid, Apollonius, Descartes, Newton, and others. In teaching Math 303–304, I hoped my own students could learn in the same way. So my course had no textbook other than the works (in English translation) of the mathematicians that we studied. In class, students presented the content of these works at the blackboard.

My notes in Part I below started out as emails to a discussion group, the "J-list," composed of St John's alumni. The dates used as section heads in this part are the original dates of composition of these emails; but I have done some editing and added some diagrams (though not yet as many as might be added for the convenience of the reader).

In the spring semester, the conversion of emails into LaTeX (so that they could be incorporated in a book such as this one) became too tedious; also I wanted to use diagrams immediately; so I started composing my notes directly in LaTeX. In Part II of this book, section titles are simply dates of classes.

A big difference between courses at St John's College and courses at METU is that the latter have written examinations. The exams that I wrote for Math 303–304 are in Appendix A.

Whether the course was a success might be judged from student comments, which I invited on the final exams; these are in Appendix B.

On the other hand, students are not necessarily the best judges of their own progress. It is also the case that one of the best and most enthusiastic students, Mehmet D., did not write me any comments; below I shall mention some of what he told me face to face. Meanwhile, I judge the course to have been successful, at least insofar as it taught students that they *could* read some of the great works of mathematics. As can be seen from their comments, some of the students wished I had just *told* them what was in those books. If the course had been simply a mathematics course, I could have done that. But the course was a *history* course, and the whole point of history is to understand what people in the past have *thought.* In saying this (and I shall say more about it below), I am following the Oxford philosopher R.G. Collingwood (1889–1943), some of whose remarks on history are in Appendix C.

My attempts to communicate to my department what I was doing with the course are in Appendix D, along with the responses of the sole person who did respond.

Appendix E consists of some notes on ancient Greek mathematics that I put on the webpage of Math 303 at the beginning of the semester.

Appendix F gives the Greek alphabet (many Greek words are quoted in the main text).

Apology

If I were to teach Math 303–304 again (which I should like to do), then I should certainly make some changes. But the practice of reading and presenting original sources, especially older ones, ought to be maintained, for reasons including the following.

Scientific history

Studying history does not mean learning to express *opinions* about what people of the past thought; it is learning what they thought. In saying this, I have in mind the distinction between opinion and knowledge expressed by the character of Socrates in Plato's *Republic* [42, II, p. 92; 506C]:

> Have you not observed that opinions (*δόξαι*) divorced from knowledge (*ἐπιστήμη*) are ugly things?*

A teacher can *tell* students what he believes Euclid thought, and the students can learn to repeat these teachings; but the teachings are only opinions for the students, if not for the teacher, unless the students test the opinions against what Euclid actually wrote.

A teacher's lectures on math history may be useful for students' *mathematics.* In *A Comprehensive Introduction to Differential Geometry* [49, p. vi], Spivak writes,

> Of course, I do not think that one should follow all the intricacies of the historical process, with its inevitable duplications and false leads. What is intended, rather, is a presentation of the subject along the lines which its development *might* have followed; as Bernard Morin said to me, there is no reason, in mathematics any more than in biology, why ontogeny must recapitulate phylogeny. When modern terminology finally is introduced, it should be as an outgrowth of this (mythical) historical development.

Spivak here is getting ready to teach mathematics, not history. It is useful for him and his readers to look at the history of the mathematics; but then that history will be adapted to the needs of the mathematics. In this case, as Spivak suggests, history becomes a myth, a kind of story. It may be an enjoyable or useful story. The story may be based on historical *knowledge* on the part of the storyteller-mathematician. But then

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* *οὐκ ᾔσθησαι τὰς ἄνευ ἐπιστήμης δόξας, ὡς πᾶσαι αἰσχραί;*

the story is not designed to share all of that knowledge. For the listener or reader then, the story—the myth—can only be a kind of *opinion,* in the sense of Plato. It is no longer history.

In *The Principles of History* [15, pp. 12 f.], Collingwood derides what he calls "scissors-and-paste" history:

> There is a kind of history which depends altogether upon the testimony of authorities . . . it is not really history at all, but we have no other name for it . . . History constructed by excerpting and combining the testimonies of different authorities I call scissors-and-paste history.

By contrast, the scientific historian will pay attention to the latest research [15, p. 35]:

> . . . whereas the books mentioned in a bibliography for use of a scissors-and-paste historian will be, roughly speaking, valuable in direct proportion to their antiquity, those mentioned in a bibliography for the use of a scientific historian will be, roughly speaking, valuable in direct proportion to their newness.

What this means for math history, I think, is that we must not treat Euclid's *Elements,* say, as the word of God or even the unaltered word of Euclid. We may well pay attention to Russo's argument in "The First Few Definitions in the *Elements*" [45, 10.15, pp. 320–7] that the obscure definition of straight line now found in the *Elements* is the work, not of Euclid, but of a careless copyist. Still, there is little point in reading Russo without reading the text associated with Euclid's name.

Experience

Most of our students will not be professional mathematicians. The experience of making sense of a difficult text, getting up in front of an audience, and talking about their understanding, will be more useful to our students than any particular piece of mathematical knowledge. Indeed, I think this is so, even for the students who *will* be mathematicians. At any rate, as I said, my own undergraduate education consisted entirely of this kind of learning. Any ability I have now as a teacher was nurtured by this experience.

Tradition

Many people derive satisfaction from their membership in a group. The group might be a political party, a nation, humanity, or the supporters of a football team. If one is studying mathematics, I suppose the best group to feel oneself a member of is the group of *mathematicians,* if not just the group of *thinkers.* By actually *reading* Euclid and his successors, we come to know that we are part of a tradition that dates back thousands of years. This point is reinforced when we consider that much of the mathematics that our undergraduates learn was created by mathematicians who had read Euclid. Most of the course Elementary Number Theory I (Math 365) at METU, for example, can be found in Gauss's *Disquisitiones Arithmeticae* (1801), of which *Wikipedia** says:

> The logical structure of the *Disquisitiones* (theorem statement followed by proof, followed by corollaries) set a standard for later texts.

This claim is not sourced, but it seems short-sighted: the statement–proof, statement–proof style of mathematical writing is found in Euclid, whom Gauss implicitly credits in his preface [27, p. xvii]:

*`http://en.wikipedia.org/wiki/Disquisitiones_Arithmeticae`, accessed June 19, 2010.

> Included under the heading "Higher Arithmetic" are those topics which Euclid treated in Book VIIff. with the elegance and rigor customary among the ancients . . . *

We may not expect our students to write as well as Gauss, even if he was only their age when he was writing; but they would do well to have Euclid as a model (and Gauss).†

Changes

Although mathematics has an age-old tradition, the subject has changed since Euclid; but this can be difficult to see. Obviously we have more theorems now; less obviously, the *spirit* of mathematics has changed. In *The Foundations of Geometry* [32], David Hilbert appears to think that, in axiomatizing geometry, he is only refining the work of Euclid. If so, Hilbert is wrong. We think today that Euclid's five postulates are not in fact sufficient to justify all of his propositions; rather, there are hidden assumptions, overlooked by Euclid, which Hilbert uncovers. Even in Proposition 1 of Book I of Euclid's *Elements,* there is an implicit assumption that two circumferences, each containing the center of the other, must intersect; this assumption is justified by no postulate. But we have no reason to think that Euclid is *trying* to uncover all of his "hidden assumptions." He is just writing down what is true.

One may say further the intersection of the circles in Euclid's I.1 is not a *hidden* assumption; the intersection is evident from the diagram. Today we think nonetheless that the existence

*The continuation of the sentence is, "but they are limited to the rudiments of the science." There has indeed been progress since Euclid.

†In my experience, the best mathematical writers among students at METU grew up in the former Soviet Union. I don't know if something in the Soviet tradition should be credited. On page 262 I quote a Soviet textbook that I used in high school.

the point of intersection should still be noted separately in words or symbols. Evidently Euclid did not think the same way. (See pp. 45 and 89 below.)

Euclid also has no notion of "non-Euclidean" geometry, so he has no need to distinguish his geometry logically from any other. His postulates, along with his demonstrations, serve as a sort of *explanation* of why his propositions are true; but there is no reason to expect the postulates and the demonstrations to provide a *complete* explanation,—if the notion of completeness even makes sense in this context. Now, although he does not seem to say so clearly, it may well be that *Hilbert's* goal was a complete set of axioms for geometry; but what could this have meant? Since Hilbert's *Foundations,* several different notions of logical completeness have been defined. Hilbert in fact succeeded in writing down a *categorical* set of axioms, in that any two geometries in which the axioms are true must be isomorphic to one another. But we can hardly say that Euclid aimed to do the same, if for him there was only one geometry in the first place.

At the beginning of "On teaching mathematics" [8], V.I. Arnold says:*

> Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
>
> The Jacobi identity (which forces the heights of a triangle to cross at one point) is an experimental fact in the same way as that the Earth is round (that is, homeomorphic to a ball). But it can be discovered with less expense.

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*I took the text from `http://pauli.uni-muenster.de/~munsteg/arnold.html` (July 30, 2010), but the original source [8] is not given there; I found it later.

> In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun).

I can't say that Arnold is right about mathematics in general. He may be right to say that *Euclid's* mathematics is physics (as physics is understood today). Again, Euclid *explains* why many geometric propositions are true. If one finds the explanations inadequate, one may probe further; this does not make Euclid wrong. Likewise, we have many explanations of the motions of the heavens—explanations by Ptolemy, Copernicus, Kepler, Newton, and Einstein. None of these explanations is wrong. Each new explanation builds on the preceding, as Hilbert built on Euclid; on the other hand, each physicist had a different project: each was looking for a different *kind* of answer to the question, "Why do the heavens appear as they do?"*

In the preceding paragraphs, I have expressed opinions about Euclid, Hilbert, and others. I might express these opinions to students; but the students should question the opinions while consulting Euclid himself (and Hilbert, and the others). It may well be that a modern mathematician misunderstands

*For example, Ptolemy wanted to know what configurations of circular motions could account for the dance of the planets in the sky. From Kepler, Newton understood that the planets moved in ellipses about the sun; Newton sought a different kind of account of this, namely a law of force.

his ancient predecessors, because his main business is to be a mathematician and not an historian. If one just wants to learn mathematics from the mathematician, that's fine; if one also wants to learn history, one should go to the source.

Proof

Many of Euclid's propositions are propositions that I learned to prove in high school, albeit from a modern textbook.* As I understood it, the purpose of my high-school course was not so much to learn those geometrical results themselves, but to learn the *possibility* of proving those results. Unfortunately students at METU seem never to get such a course, either in high school or at METU (see p. 59). We do teach proof; but at the same time we are teaching modern mathematics, and this complicates things. I count Descartes as modern. Descartes gives us a method of great power, which students begin learning in their first semester at METU, in Analytic Geometry (Math 115). However, it is difficult to understand the method's power of *proof.*†

Using analytic methods, how would we prove that the base angles of an isosceles triangle are equal? Given such a triangle, we can set up a rectangular coordinate system in which the vertices A, B, and C of the triangle are respectively $(0, a)$, $(-b, 0)$, and $(c, 0)$, where a, b, and c are all positive (Fig. 0.1). Then $AB = AC$ if and only if $a^2 + b^2 = a^2 + c^2$, that is, $b = c$ (since both are positive). In this case, the angles at

*I didn't much like the textbook, which was [57]. I wanted to read Euclid, and did so, first on my own, and then at St John's.

†See §A.5 for an exam that required application of Descartes's analytic geometry, as well as Newton's conception of quadrature. Most students performed very poorly on this exam; later I discuss what to do about this.

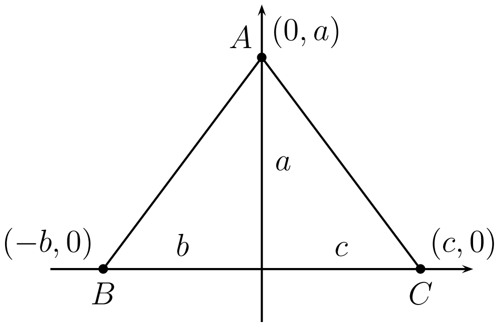

Figure 0.1. "Analytic" proof of Euclid's I.5

B and C have the same cosine, namely $b/\sqrt{a^2+b^2}$, so the angles are equal. Fine; but this argument uses notions not found till page 133 of the analytic geometry text [34] used at METU; even then, the text just assumes familiarity with cosines, when full knowledge of these will not come till a later course of mathematical analysis.

By contrast, for Euclid, the equality of the base angles of an isosceles triangle is only Proposition 5 of Book I of the thirteen books that make up the *Elements*. Notwithstanding the "hidden assumptions" mentioned above, I don't know anything better than Euclid's "synthetic" geometry for giving students a notion of what is a sound proof.

Thrills

I just mentioned the power of Descartes's analytic geometry. It is a thrill to learn this geometry from Descartes himself. The thrill is worth sharing with students; but it does not come cheap (my colleague's dismissive comments on p. 250 notwithstanding). One needs to have read Descartes's predecessors, and to have read them faithfully—*not* translated into modern, symbolic, *Cartesian* language. But textbooks like Boyer [11] present the old work in just this anachronistic way.

Discoveries

It may happen that new mathematics comes out of taking old mathematics seriously. I can only offer my own example: a paper about the logic of vector spaces [41], directly inspired by reading Euclid and Descartes.*

Coverage

The wonderful new *Princeton Companion to Mathematics* [29] contains short biographies of 96 mathematicians, in chronological order. The first 14 mathematicians listed are:

(1) Pythagoras,
(2) Euclid,
(3) Archimedes,
(4) Apollonius,
(5) Abu Ja'far Muhammad ibn Mūsā al-Khwārizmī,
(6) Leonardo of Pisa (known as Fibonacci),
(7) Girolamo Cardano,
(8) Rafael Bombelli,
(9) François Viète,
(10) Simon Stevin,
(11) René Descartes,
(12) Pierre Fermat,

*The main algebraic result is that if we have a vector space of dimension greater than n, then we can enlarge the scalar field so that the dimension of the space is reduced to n, while every set of n vectors that are linearly independent over the original scalar field remain independent over the new field. Logically then, model-theoretically, the theory of n-dimensional vector spaces over algebraically closed fields is, in the appropriate signature, model-complete. Casting all humility aside, I quote from the anonymous referee: "The paper is well-written and very interesting. The structures are indeed basic, yet I found several results which surprised me, and the technical proficiency with which things are handled makes publishing the presentation worth-while. For example realizing the geometric idea of Descartes, while taking care to make all formulae existential, is an example of the added value of the paper. To me personally even the fact that the scalar field can be recovered from the parallelism predicate was new."

(13) Blaise Pascal, (14) Isaac Newton.

In my course, we read works of seven of the mathematicians on this list. (There is no extant text by Pythagoras. In one class I lectured additionally on Archimedes.) We also read two other mathematicians, namely Thābit ibn Qurra and Omar Khayyám; but we could have dropped the former, in line with the suggestion of Ali in §B.2 (p. 236). It is indeed a shame not to read any of the remaining 82 mathematicians on the *Princeton Companion's* list. But there just isn't time to read many more. One could read the work of many mathematicians in a source book like Smith's [46] or Struik's [50], but I think the coverage would be too superficial to be of much value.

If it is desired, then Newton's contemporaries (such as Leibniz) and successors can be studied in the courses that cover their work. About courses I have taught at METU, I can say that Gauss can be read in Elementary Number Theory I (Math 365), while Set Theory (Math 320) and Introduction to Mathematical Logic and Model Theory (Math 406) can make use of van Heijenoort's anthology of mathematical logic [55]. It just does not seem fair to me to use a course like Math 303–304 to teach students about mathematicians whose work they do not have time to *know.*

If one wants a royal road to a view of the grand sweep of mathematical history, one can read Struik's *Concise History of Mathematics.* However, one might be uneasy with the author's materialistic approach. Struik writes for example [51, Ch. V, §6, p. 82 and Ch. VI, §1, p. 93]:

> The main line of mathematical advance passed through the growing mercantile cities under the direct influence of trade, navigation, astronomy, and surveying. The townspeople were interested in counting, in arithmetic, in computation. Som-

> bart labeled this interest of the fifteenth- and sixteenth-century burgher his *Rechenhaftigkeit.** Leaders in the love for practical mathematics were the "reckon masters," only very occasionally joined by a university man, able, through his study of astronomy, to understand the importance of improving computational methods.
>
> The rapid development of mathematics during the Renaissance was due not only to the *Rechenhaftigkeit* of the commercial classes but also to the productive use and further perfection of machines.

This explanation of the development of mathematics is perhaps correct; but it is hardly complete. The development of mathematics is due, first of all, to *mathematicians* (who may or may not be "university men"). They may take their inspiration from various sources; but those sources do not *cause* the mathematics to be created. This is a point worth making in a course, and it is a point made by the practice of reading *mathematicians.* Struik may not disagree. In the introduction of his history, he writes:

> The selection of material was, of course, not based exclusively on objective factors, but was influenced by the author's likes and dislikes, his knowledge and his ignorance. As to his ignorance, it was not always possible to consult all sources first-hand; too often, second- or even third-hand sources had to be used. It is therefore good advice, not only with respect to this book, but with respect to all such histories, to check the statements as much as possible with the original sources. This is a good principle for more than one

*[Struik's note:] W. Sombart, *Der Bourgeois* (Munich and Leipzig, 1913), p. 164. The term *Rechenhaftigkeit* indicates a willingness to compute, a belief in the usefulness of arithmetical work.

reason. Our knowledge of authors such as Euclid, Diophantus, Descartes, Laplace, Gauss, or Riemann should not be obtained exclusively from quotations or histories describing their works. There is the same invigorating power in the original Euclid or Gauss as there is in the original Shakespeare, and there are places in Archimedes, in Fermat, or in Jacobi which are as beautiful as Horace or Emerson.

Possibilities for the future

I would make some changes in teaching Math 303–304 again. Here are some notes about what might be done.

If students are going to make presentations, they must prepare for these conscientiously, with the understanding that a poor presentation will disappoint not only their teacher. *Classmates* must challenge students who try to fake their way through a proof. Such challenges happened occasionally in my class (see for example p. 69); I wish I could encourage students to make more of them, or (better) convince the speakers not to try to fake their way. (See pp. 88, 98, and 106, and §5.5 [p. 171], for some problematic days.)

Some formal measures might be of help. I did learn all of my students' names; but I found out too late that they didn't always know *one another's* names. I sometimes tried arranging the desks in a semicircle (see pp. 60 and 173). I am told by Mehmet (whom I mentioned above) that what I really must do is *grade* the students on their individual presentations. Mehmet is not a student who needs such a goad, but (if I understand him) other students do need prodding by the threat of low marks. In this case, I can only hope that what students first do for marks, they may later do for their own satisfaction. As it was, I did tell students that they got credit for *attending* class; I did not say that students would be graded on the *quality* of their attendence and participation.

Also (suggests Mehmet), students should know many weeks in advance what they will be presenting. This should be possible, now that I know (from this very log) at what pace the course can proceed. Mehmet did think the practice of reading original sources like Newton should be continued.*

Classes proceeded more slowly than I expected, sometimes because students had indeed not conscientiously prepared for them. If one wants to cover more material, one can skip some propositions in class, while holding the students responsible for learning them independently. Students might still work together, as Ece suggested; see §B.2 (p. 234). Still, it should be noted that, though at the beginning of Math 304 I assigned presentations to pairs or triples of students, the students generally didn't work together.

The teacher could compromise his principles and make some presentations himself. Indeed, as I noted above, I did this with Archimedes. I did it too with Book V of the *Elements,* on proportion (see p. 62), and I should have done it more; here, understanding the mathematics is hard enough, even if one is not trying to learn the mathematics straight from Euclid. The final exam of Math 303 showed that students had *not* generally learned Euclid's definition of proportion (the one that must have inspired Dedekind's definition of the real numbers [16]).†

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*Mehmet took both semesters of Math 303–304 and is now [June 2010] going to study for a doctorate in physics at Yale University.

†Russo [45, pp. 46 f.] ridicules historians like Heath, who are impressed that Euclid could have "anticipated" Dedekind's theory of irrational numbers. Euclid didn't anticipate Dedekind; he *taught* Dedekind, who read him in school.

It was hard for the students not to have much sense of what would be on exams. I didn't have much sense myself, when I started the course. Nonetheless, in the first semester, students generally impressed me by their understanding on exams; but in the second semester, they disappointed me. I was quite pleased with the problems I wrote on the last two exams of Math 304; but in another year, some such problems should be worked out with students in class.

In lectures in the second semester, I compromised and *stated* for the students the results from Apollonius that we would need for Newton. Proofs of some of these results did however end up as the exam problems just mentioned. Again, it would be better to make proofs of *all* of these results more clearly a part of the class, either in lectures or in homework. Proofs can use the streamlining that Descartes makes possible, at least if the point is to be able to read Newton's *Principia.* (In Math 304, I told the students out loud that problems like those on the second exam could show up on the final; but the students seemed not to do anything with this warning.)

Anthologies like Katz [35] are useful for identifying the old works of mathematics that may be worth reading. However, it may be misleading to see a brief excerpt out of context. It would be desirable (where possible) to consider an anthology's selections in the context of the larger works from which they have been taken.*

Unfortunately most students of Math 304 probably will not have taken Math 303. Therefore it may be better to divide the contents of the course not chronologically, but thematically, perhaps with geometry and analysis in one semester, number

*I have therefore asked the METU library to order some of these larger works.

theory and algebra in the other. The former could start with Book I of Euclid's *Elements;* the latter could leave this book as background reading, but start seriously with Book II.

I had originally thought of finishing Math 304 with Lobachevsky, but there was no time, and anyway most of the students had not read Euclid, because they had not taken Math 303. In a rearrangement of the course, Lobachevsky could be accommodated somehow. On the other hand, Lobachevsky is number 31 on the (chronological) list in the *Princeton Companion;* we would skip a lot of great names to get to him.

D. P.
Ankara
July 30, 2010[a]

[†]Minor editing, June 20, 2011, and later.

Part I.

Fall semester

1. Euclid

1.1. Sunday, October 4

This is about a course I am teaching now: a course in which students read and present Euclid in more or less the St John's style. We have had three hours of class so far, and I am excited to think that the course may work out as I have hoped.

My own memories of the first-year math tutorial at St John's are dim. Possibly Johnnies* are better prepared to read Euclid, precisely because they haven't spent two or three years studying modern mathematics as my students now have—and because Johnnies have come to college expecting to read old books.

I am teaching at Middle East Technical University, in the capital of Turkey. The course I am writing about now is a third-year course that comes with the rather pretentious title "History of Mathematical Concepts I." I didn't ask to teach it. But students wanted the course to be offered, and a few weeks ago, one of the assistant chairs of our department asked Ayşe (my colleague and spouse) if she would teach the course. She didn't want to, but said that I might. Eventually I was offered the course.

I didn't want to teach the course as my colleagues had done

*As noted on p. 11, these remarks were originally addressed to St John's College alumni, who call themselves Johnnies.

in the past: from a textbook like Boyer's *History of Mathematics* [11]. But I realized that the course could be an opportunity to read original texts with students. I decided to take it on.

Unfortunately our students are used to skipping class. I think they may pick up this idea from high school. High school in Turkey does not prepare students for the national university entrance exam; the students take special lessons on evenings and weekends for this. There the students learn all the tricks that their regular teachers don't tell them. So what's the point of spending time with a regular teacher at all?

In my other courses at METU, I have not required attendance. If students want to study on their own, that has been fine; all that matters is their performance on exams. But in a course where the whole point is to read and discuss Euclid, this won't do.

Students started registering for courses this semester on Wednesday, September 23. On that day, my course was open to third-year students, with a capacity of 30. In the afternoon, 23 students had registered. Then I sent out an email to all math students, warning them of the unusual nature of the course. I threatened them with failure if they did not come to class. The authorities had not limited course capacity as severely as I had wanted, so I tried to scare away uninterested students. On Thursday, the course was open to fourth-year students, and the capacity was raised to 40. This capacity was reached, but now only 17 of those students were third-year. It seems I had driven off six students.

In the following week—last week—I met my class twice. On the first day, Tuesday, 24 students showed up. On the second day, Friday, only 18 showed up, though four of them had not come on Tuesday. So I have seen a total of 28 students, out of 40 who had registered (plus three more who couldn't, but still wanted to come). This is probably typical. Students don't have to commit to courses till the coming week, "add-drop week" (when I shall make sure that those three extra students can register, if they still want to).

On Tuesday, the first day of class, at first I didn't speak of definitions, postulates, and common notions. I just proved Euclid's Proposition I.1 (to construct an equilateral triangle), and then I asked what we had assumed in constructing the triangle. Thus we recognized a need for Postulates 1 and 3. One student observed what is famously missing from Euclid: we need to know that the two circles in the construction intersect. (I don't hold this to be a flaw in Euclid. As I think I have learned from Mr Thomas on the J-list, the flaw is to think that Euclid is trying to present an axiomatic system as we understand such things today.)

We went on to prove and discuss Propositions I.2 and 3.

Unfortunately here in Ankara one cannot order textbooks and expect students to buy them. One reason is that our department does not tell us what we are teaching until it is too late to order books. Another reason is that books are expensive, and as long as the library has a copy, students will have it photocopied. As for Euclid, the library seems to have lost some volumes of the Dover edition of Heath, and the library hasn't bought the Green Lion edition yet. So I have pointed the students to several web editions of Euclid. I could make my own copies of the Green Lion or Dover edition available for photocopying, but I won't.

Perhaps the recent Fitzpatrick translation of Euclid is the most useful, if only because the author has put a `pdf` file on the web.* However, I recall that Mr Thomas had some criticism

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*`http://farside.ph.utexas.edu/euclid.html`

of this edition, or at least was dubious about the reliability of the Greek text that accompanies the translation. My perusal suggests that Fitzgerald is more literal than Heath, but his footnotes may be misguided.

On Tuesday, on the web* as well as in class, I hoped I had been clear enough about what was expected from students. On Friday however, it appeared that few students had got around to actually reading Euclid; or perhaps they were shy about admitting it. One student agreed to present Proposition 4 (what English-speakers may learn as SAS); but then it transpired that she had read only **my** account of this proposition, which I had also put on the web, perhaps by mistake.

For Proposition 5, nobody was initially forthcoming. I drew a triangle ABC on the board, with $AB = AC$, and invited somebody to try to prove the equality of angles ABC and ACB. This was a useful exercise, for me at least. One student came up and drew a circle whose center was A and whose circumference contained B and C. Tolgay tried to argue that the base angles of the triangle subtended equal arcs of the circle, or were inscribed in equal arcs, or something like that.

The state of Tolgay's mathematics may be like what I imagine mathematics to have been before Euclid. Tolgay understands that we can prove some propositions from other propositions. But he has no clear notion of a systematic development, from a few basic principles, of a whole body of mathematics.

When I visited St John's as a prospective student, my guide took me to his dorm room, where he and his friends told me excitedly that whereas in high school you were **told** that things were true, at St John's you **proved** them. So these were my

*`http://metu.edu.tr/~dpierce/Courses/303/`

people. In fact my own high-school geometry course had been rigorous,—so much so that I understood the whole point to be not isosceles triangles and parallelograms, but proof itself. Still, during that course, I obtained a copy of Euclid, and I wished we could read this instead of our regular textbook.

After geometry, I had a two-year course of calculus, where we proved everything from the axioms for a complete ordered field (the so-called real numbers). As I understood it, this was what mathematics was all about.

Our students at METU are among the best in Turkey, and they have learned to do some math problems that I haven't a clue how to solve. But apparently it's hard to ask about proofs on a multiple-choice university entrance exam. In any case, our students don't seem to come to us with much notion of proof. We have a first-year course that is supposed to instil such a notion; but it is also supposed to teach about "linear orderings" and "equivalence classes" and various other modern abstract notions. I have thought that students might be better served by a course of reading Euclid.

The student Tolgay at the board, trying to prove I.5 with a circle,— he can apparently think creatively, but if after two years of university mathematics he can't catch on to what Euclid is about, even just from attending an hour or so of my class, then I think there may be something wrong with our department's program.

Maybe my criticism is premature. In any case, I suggested that Tolgay was trying to use some propositions that were indeed correct, but that we had not proved yet.

Another student [name forgotten] came forward and tried to prove I.5 by drawing through A a straight line parallel to BC. I pointed out that as yet we knew nothing about parallel lines.

Finally it appeared that somebody had read Euclid. Proposition 5 was presented faithfully by Ali, who on Tuesday had transferred the `pdf` file of Fitzgerald's Euclid from my flash drive to his.

I gave Pappus's proof of I.5, which is much easier to write down than Euclid's, but perhaps harder to believe. (Triangles ABC and ACB are equal in all respects, by I.4.) Students seemed to like this proof, including Ali. I asked whether Euclid might have known the proof.

Proposition 6 was also presented *à la* Euclid, this time by a young woman who had dropped out of my set-theory class last semester because her father was dying. I learned about the death after the course, when Elif sent me an email thanking me for letting her pass anyway. I had been quite lenient that semester, because of another student, who had had to undergo treatment for leukemia. Elif wrote that she hoped to do better in another course of mine; perhaps this semester she intends to fulfill that pledge.

For Proposition 7, Cihan came forward to give the proof. He was one of the three students who had asked me to enlarge the capacity of the course so that he could join. Cihan seemed to have read the Euclid, though he got confused. I pointed out that Euclid's proof covered only one case: another arrangement of the points was possible for which the proof wouldn't work. I left consideration of this case as an exercise.

I think I myself did Proposition 8.

For Proposition 9 (to bisect an angle), Cihan eventually came forward again, though seemingly without any notion of Euclid's construction. He gave his own argument, assuming that a straight line could be bisected (as in Proposition 10). Ahmet came forward with a correct method of bisecting a straight line, but he had trouble proving it until Ali came up to help.

First Ali gave some advice from his seat, in Turkish. I let the Turkish discussion go on for a bit, then pointed out (in Turkish) that not everybody knew Turkish. I didn't mean only myself. There was a British student in the class, here just for the semester. There's also an Albanian student, though he may have learned Turkish. There is a student from Azerbaijan, but Azeri and Turkish are mutually comprehensible.

By this time it was late on Friday afternoon, and our two hours were up. I had pointed out to the sleepy students that I hadn't chosen the schedule. I decided reluctantly to take volunteers for the next few propositions, to be presented next Tuesday.

1.2. Thursday, October 8

Tuesday's class is from 16.40 to 17.30. On Tuesday of this week, I went to the classroom ten minutes early, to be able to get on with the business of learning people's names. Two students were there. Yunus asked how many exams there would be, and I repeated what I had written on the webpage: one midterm and one final. He asked what would be on the exams, and I said I would ask for proofs, perhaps of propositions from Euclid, perhaps of others. He asked what they (the students) could use in the proofs, and I said they could use whatever Euclid used. He asked whether that meant they could bring a list of Euclid's propositions to the exam, and I said no. I said students should know the Greek alphabet. Yunus asked where they could get the alphabet, and I repeated what I had added to the webpage that day: They could get it from *Wikipedia* for example, or they could download a page prepared by me.

I had brought a printout to class, so I gave it to Yunus.

It is tedious to read the last paragraph, but it was tedious to go through this dialogue with Yunus in the first place. The system trains students to ask such questions; and anyway I **am** going to have to assign letter grades at the end of the semester.

By the time class was supposed to start, one other student had shown up. She said many students were coming from another class, which was then being held in a building far away, because of the ongoing renovations in our department's building. I asked about that other class, and then I realized it was my own spouse's class! Ayşe assured me later that she had ended class on time. But you know, students don't feel like rushing from one class to another.

Soon more students came, and the presentations of propositions started.

As I recall my own freshman math tutorial with Mr Kutler in Annapolis, propositions were not preassigned to students. I don't recall any problem finding volunteers on the spot, although this may be because people like me were prepared to volunteer if nobody else was. The year before, when I visited a math tutorial as a prospective student, a volunteer was **not** forthcoming for Proposition N. The tutor then closed his eyes and brought his pencil down on the list of students. The student so picked asked nervously, "Could I do Proposition $N + 1$ instead?" I suppose he knew he had to present something, and he couldn't be prepared for everything, so he prepared for that proposition.

I had hoped the class I am teaching now could be like the one I was a student in, or at least like the one I was a prospective student in. But as I suggested in my report on last Friday's class, I gave up on that idea pretty quickly. I took volunteers on Friday for the following Tuesday's propositions.

On Tuesday, therefore, the exchange-student Jeremy came forth with I.11 all prepared. He started writing out the statement of the proposition, in what seemed to be a direct quote. I worried that he was just going to quote the proof as well, but he didn't. I raised several questions, during the proof and afterwards. "How do you know point E exists?" "Why doesn't Euclid just draw a circle to find E, rather than appealing to Proposition 2?"

Things continued in this way. I raised questions, trying to suggest the kind of critical approach that I hoped the students themselves would take. But I suppose it's hard for them to get critical about the seemingly basic propositions we are going through.

Besmir proved I.12. Then Yunus proved I.13, that a straight line set up on another makes angles equal to two right angles. He did it more tersely than Euclid, and I wondered why Euclid's approach had an extra complication. Yunus argued that, if the straight line AB is set up on CD [so that B lies on CD], and EB is set up at right angles to CD, [and E is on the same side of CD that A is on], then

$$CBA + ABD = CBE + EBD = 2 \text{ right angles.}$$

(See Figure 1.1.) But Euclid argues (in prose that one might

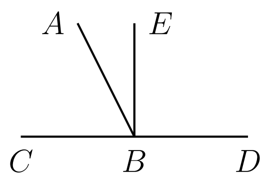

Figure 1.1. Euclid's I.12

rewrite as follows):

$$\begin{aligned}
CBA + ABE &= CBE, \\
CBA + ABE + EBD &= CBE + EBD, \\
ABE + EBD &= ABD, \\
CBA + ABE + EBD &= CBA + ABD, \\
CBA + ABD &= CBE + EBD = 2 \text{ right angles.}
\end{aligned}$$

Why such length? I don't know, unless, for Euclid, the sum of angles CBA and ABD is not itself an angle, so it cannot be immediately identified with the sum of CBE and EBD.

Indeed, I have seen it said that one sign of Euclid's greatness is his **not** trying to treat angles as if they were the same sort of magnitudes as straight lines. Today one might say that the sum of right angles CBE and EBD is a "straight angle," whose measurement is 180 degrees. Then "obviously" the sum of CBA and ABD is the same. But this is not obvious for Euclid, and rightly so.

It is a failing that Heath does not comment on I.13, except for a remark on translating one clause. I don't know if this is Heath's failing, or a failing of other commentators whose work he reviews in his own notes.

Damla, Friday's volunteer for I.14, did not show up for class on Tuesday. Cihan stepped up to prove the proposition. In the proof by contradiction, he established

$$\begin{aligned}
CBA + ABE &= 2 \text{ right angles,} \\
CBA + ABD &= 2 \text{ right angles;}
\end{aligned}$$

then he concluded

$$CBA + ABE = CBA + ABD.$$

I asked how this was justified, and he mentioned Common Notion 1: Things equal to the same are equal to each other. I pointed out that we don't know that **these** two right angles are equal to **those** two right angles. Somebody pointed out that we had Postulate 4.

It has been two days since the course, and I don't remember who brought up Postulate 4. Seçil presented I.15 ("vertical angles are equal"), and then I took volunteers for Friday's class. I spent half an hour after class talking with one student and then another, about the Heath translation versus the Fitzgerald, and about how ancient mathematics may differ from our own.

1.3. Friday, October 9

My course is an "elective," and students who don't like it can take another. (For several electives, such as Ayşe's on graph theory, enrollment is maxed out. But a third-year elective on Lebesgue integration has only 18 students registered; a couple of fourth-year electives have less than 15 students; another course will be closed for lack of interest.)

On this last day of "add-drop" week, I am down to 22 registered students. Of those 22, there are 6 whom I have never seen in class. Another came to the first class only. Of the remaining 15, all but two have come to every class; the other two came only to the most recent class, but seem to be serious about the course: they are among the volunteers to present propositions today.

In short, I seem now to have 15 interested students. After today's class, each of these 15 will have been up to the board at least once to present a proposition. Those other students who didn't want to do this will have dropped out.

In my very first seminar at St John's, I was a bit surprised when a tutor launched into an opening question without any preliminary remarks about how things would be done. But that was fine of course, and I guess I'm following that model today.

I plan to cover all of Book I, and then pick and choose (in St John's fashion, as I recall it). We should cover the theory of proportion. The students all know the "Euclidean algorithm" for finding greatest common divisors; it would be good for them to see this in its original presentation.*

After that, I don't know. We should do something of Archimedes and Apollonius. (I note that Euclid, Archimedes, and Apollonius seem to be the main sources of examples in Reviel Netz, *The Shaping of Deduction in Greek Mathematics* [39], the book that Mr Thomas recommended.)

At the Nesin Mathematics Village in the summer of 2008, I presented much of Book I of Apollonius in 12 hours of lectures. This would translate into four weeks of my present class; but it would be too fast (as it was in 2008), and not in the right spirit. Really students should be presenting propositions, even if this seems to slow things down.

Next semester, I am likely to be assigned "History of Mathematical Concepts II," which is supposed to start with the Renaissance. Ayşe suggested I could spend all of the present semester on Euclid, and all of next on Apollonius, regardless of what the course catalogue says; but I don't think I'll do this.

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*We eventually skipped this in class, however.

1.4. Saturday, October 10

On Friday, October 9, Euclid class began with Ahmet's presentation of I.16: in a triangle, an exterior angle is greater than either of the opposite interior angles. Not all students were present at the beginning of class. I had come to class with various things to say, but I could say them any time. I let Ahmet be the first speaker, so he could have the experience of seeing latecomers walk in while he talked (and so that they would see that they were interrupting one of their classmates).

Ahmet and I are old friends: he took model theory with me last fall and set theory last spring, and he used to ask challenging questions after class. He is a double major in math and philosophy. He and another undergraduate named Burak inspired me to offer a reading course this semester, in addition to the two courses I am normally assigned. (We intend to read together the late Paul Cohen's book [13]—based on his lectures at Harvard—on his proof of the independence of the Continuum Hypothesis.)

Ahmet expressed the equality of two lines by writing

$$|AE| = |CE|.$$

Other students had used this notation before. I asked what the vertical bars meant, and of course Ahmet said that they denoted taking the **lengths** of the lines. If I understood his point, he said that we couldn't do the math unless we had the abstract notion of length. I observed that, as far as I knew, Euclid didn't refer to length as such; he just used ordinary language, saying AE was equal to CE. Nobody, including myself, recalled that, in Definition 2, a line is "breadthless length." Now, there is an argument (by Lucio Russo, in *The Forgotten Revolution* [45]) that Euclid didn't write this or any

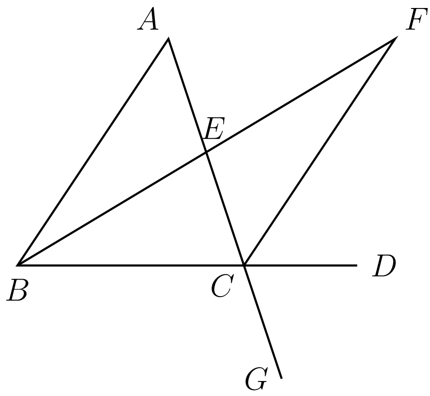

Figure 1.2. Euclid's I.16

other definition of line, but in any case, Definition 2 says, not that a line **has** a length, but that a line **is** a length.

In the book recommended by Mr Thomas called *The Shaping of Deduction in Greek Mathematics* [39], Reviel Netz observes that the **words** of Euclid and other ancient mathematicians do not completely determine the diagrams. I assume now that the interested reader can look at the diagram in Heath's translation (now reproduced as Figure 1.2). After Ahmet had proved angle ACD greater than angle BAC, he observed that, by the same construction, BCG is greater than ABC, while BCG is equal to ACD by I.15. I said it wasn't necessary to repeat the construction, but nobody seemed to get the point until I spelled it out: since we have proved $ACD > BAC$, we have proved the general statement of the proposition, another instance of which is the inequality $BCG > ABC$. No further proof is necessary.

When Ahmet was finished, before I could make any of the general remarks I had prepared, Mehmet stood up to continue with I.17: any two angles of a triangle are together less than two right angles. I let him proceed. (Mehmet, Ahmet, and

Burak were by far the best students in set theory last semester. Mehmet is majoring in physics as well as math.)

It may have been during Mehmet's presentation that a student whom I hadn't seen before, Rashad, mentioned 180 degrees. Perhaps he thought I.17 was obvious, since all three angles of a triangle add up to 180 degrees. There was some laughter when I pointed out that we didn't know anything about degrees; perhaps the other students had got used to hearing me say such things.

After Mehmet, I introduced the parts of a proposition that are spelled out by Proclus in his commentary [43] on Book I of the *Elements.* The relevant section of Proclus is quoted in the introduction to the Green Lion edition of Heath's Euclid (though unfortunately without a page number—it's 159 in the cited translation). So recent Johnnies should know the parts of a proposition. Netz gives them in his book as well, where they are called

(1) Enunciation (*πρότασις*: what is to be proved in general terms);
(2) Setting out (*ἔκθεσις*: the "givens" as labelled in the diagram);
(3) Definition of goal (*διορισμός*: the "to prove");
(4) Construction (*κατασκευή*: additional straight lines and so forth that are needed in the proof);
(5) Proof (*ἀπόδειξις*);
(6) Conclusion (*συμπέρασμα*: a repetition of the enunciation, and what Heath replaced with "Therefore etc.").

In writing (5) on the board, I asked Ahmet whether he had encountered the word "apodictic" in a philosophy course; he seemed to find the word vaguely familiar.

In listing the six parts, I just wanted to be clear that the things we call "propositions" have a definite form, a form which, for the sake of the reader, the writer might choose to follow.

By the way, Netz in effect points out that our use of the word "proposition" is an instance of metonymy. Properly the proposition is only the enunciation: part (1) above. Netz argues that, for the Greeks, the "metonym" for the whole six-part package was not the enunciation, but the diagram. Now, the diagram is not one of those six parts. One might think that the diagram is like the **sight** or **look** of a person, while the six written parts are the **voice** of the person. In any case, Netz's argument is tenuous, or else I am reading too much into it.* He observes that, even when the same diagram **could** be used for two propositions, it almost never is. In "translating" the *Conics,* Heath [1] mutilates Apollonius precisely by making one diagram fit many propositions.

In class, later presentations of propositions seemed to be influenced a bit by Proclus's list of parts. But I saw then that I had a task for the future: to convince students not to write down the "definition of goal" without being clear that it hasn't actually been proved yet. The students tend to write formulas without writing any words to explain their interrelations. Proofs should be persuasive prose compositions; but the students get little or no practice in writing in school. (Remember, the university entrance exam is all multiple choice.)

In particular, in her proof of I.18 (in a triangle, the greater side subtends the greater angle), Özge used some of the terminology from Proclus. Next up was Mürsel, who had sat in only on the previous class before deciding to register for the course. His argument was quite detailed in a good way. But he kept looking at me, sitting at the side of the room, until I

*Editing these remarks later, I don't remember why I thought Netz's argument tenuous.

reminded him that I wasn't the only student in the class. He had a soft voice, and I think it was he whom I asked, "Do you think your classmates can hear you?" Those classmates said "No!"

Break time was coming up. I stepped up to show that, as I gather from Netz, Euclid tends to be mistranslated in English. The "setting out" of I.1 is not "Let AB be the given straight line" but rather "Let the given straight line be AB":

Ἔστω ἡ δοθεῖσα πεπερασμένη ἡ AB.

Netz argues that Euclid is not **creating** a straight line whose endpoints are defined to be A and B; rather, there is already a straight line; it is "given"; it is there on the diagram (Figure 1.3) and its endpoints are those that you see near the

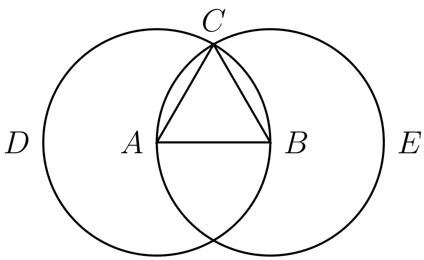

Figure 1.3. Euclid's I.1

letters A and B.

What this means to me is that the recent translator Fitzpatrick is wrong to say in a footnote to I.1, "The assumption that the circles do indeed cut one another [at C] should be counted as an additional postulate." That the circles do cut one another is too obvious to need postulating: you can see right there in the diagram that the circles cross. In our modern ideal, a proof proceeds by mechanical application of formal rules to strings of written symbols. A proof is a list of such strings, with nothing to do with any diagram. But Euclid was not trying to write such proofs.

So Fitzpatrick, like Heath, seems to have overlooked one feature of the Greek. But perhaps he did well to translate Greek perfects as English perfects: "Let the circle BCD . . . *have been* drawn." The diagram is not being constructed as the reader reads; the diagram is already there, it already **has been** constructed. (And since the reader is probably reading a scroll, the reader never needs to flip a page. Again I owe these observations to Netz. But in the Green Lion edition [23] of Heath's Euclid, diagrams are repeated on every spread where they are needed. Is anybody here young enough to have benefitted from this feature? The luxury of a scroll, the convenience of a codex! How's that for an advertising slogan?)

During the break, I noticed a sticker on the new classroom windows that said *ısı yalıtımlı çift cam.* This meant "heat ____ double glass," and one can guess the meaning that fills in the blank, but I didn't recognize *yalıtımlı.* I guessed that it came from a verb *yalıtmak,* which might in turn be the causative form of a verb **yalmak.* Melis and Ali told me I was right on the former point, wrong on the latter. *Yalıtmak,* said Ali, meant "insulate" or "isolate." Knowing that even my spouse confuses these two English words, I wrote them on the board. Jeremy from the UK explained the distinction. I observed that *insula* was island in Latin. I recalled that, in ancient times, way out on the tip of what is now Turkey's Datça Peninsula, there was a city called Knidos. We once talked on the J-list about the Aphrodite of Knidos. The Knidians tried to "isolate" or "insulate" themselves by cutting a canal across their isthmus, making their home an island; but they failed. (The story is in Herodotus. In class I observed that *insula* appeared in "peninsula," but forgot that the Turkish word, *yarımada,* was also literally "half-island.")

We were still in the break, and not all students had returned to the classroom, but I couldn't wait to talk about Greek imperatives, particularly in the third person. As I had reviewed in Smyth's *Greek Grammar* [48] in the morning, there are three kinds of Greek imperatives: present, aorist, and perfect. Moreover, the personal endings have different forms in active and passive voice. Turkish has just one kind of imperative, and distinctions of voice are handled in a different part of the verb. (I didn't get into the middle voice, but Turkish as well as Greek might be said to have one.) Actually, classical Greek apparently forms its active perfect imperative periphrastically, as a participle + "let it be" (*ἔστω*). Turkish uses a similar construction for a perfect imperative (as in *Geçmiş olsun* "may [your trouble] have passed"). As in Greek, the Turkish for "let it be" is one word (*olsun*).

By this time class was officially going, and I emphasized my main point: not that students should know all of the Greek imperatives, but that Euclid used one of them in particular, the perfect, to talk about things that had already been constructed. Nonetheless, Elif asked me to clarify the distinction between the present and aorist imperatives. I hadn't actually used those terms, but had just written down a sort of paradigm (Figure 1.4)* based on *γράφω*. As for the distinction, I just repeated briefly what I had gathered from Smyth:

(1) *γραφέσθω* let [it] be drawn [generally]
(2) *γραψάσθω* let [it] be drawn [now]

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*My Greek is so rusty that I am not entirely confident of these forms, though Smyth does give the perfect middle/passive of *γράφω* as a paradigm in his ¶ 406.

	active	middle	passive
present	*γραφέτω*	*γραφέσθω*	
aorist	*γραψάτω*	*γραψάσθω*	*γραφήτω*
perfect	*γεγραφὼς ἔστω*	*γεγράφθω*	

Figure 1.4. Greek imperatives

(3) *γέγραφθω* let [it] have been drawn, *çizilmiş olsun*

Still, Taner asked what the point of learning Greek was, since math was hard enough in one's own language, and harder still in English. I said I just wanted him to know the alphabet and to recognize the Greek words, like *γράφω*, that are the source of our mathematical vocabulary. I passed along the rumor that I had heard on the J-list from Mr Billington, who had heard it from a British woman [Deborah Hughes Hallett] who once taught in my department here in Ankara (though well before my time): math students do better if they have learned the Greek alphabet. I pointed out in Turkish that, unfortunately, I didn't know much Turkish, but anyway, English was spoken at METU. In English I said it would be good if somebody would translate Euclid into Turkish; but the translator should work from the original Greek, not the English.*

Taner was supposed to present I.20 (two sides of a triangle are together greater than the third), so perhaps he was nervous. He had come to my office earlier in the day, not having consulted the course webpage or actually looked at Euclid yet. He asked what the point of the course was, and I told him as

*Ahmet Arslan, translator of Aristotles's *Metaphysics* into Turkish [7], confesses in his preface that he does not know Greek: he used French and English translations for his own work.

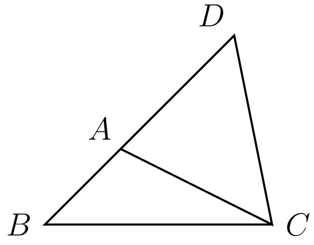

Figure 1.5. Euclid's I.20

best I could.

In class, his presentation showed understanding, but was rushed, and people besides me raised questions. Ali asked Taner to write down what he was proving. (See Figure 1.5.) In his haste he got it wrong, instead of

$$BA + AC > BC$$

writing something like

$$BA > BC.$$

For a while I thought maybe he meant that we could **assume** what he had written down, and this would have been correct, as far as it went. We got things straightened out in the end. As Taner returned to his seat, he apologized to me for his poor English; actually his English was good.

At some point in the class, I went back to talk about I.16 (Figure 1.2). Some commentators say it must use a hidden assumption, since it doesn't use the parallel postulate, and yet it fails on the surface of a sphere. I observed that the proposition fails if the point F ends up below BD (that is, on the other side from A). Ali said that couldn't happen, because then BF would not be straight—that is, it would not "lie evenly with the points of itself." I suggested that, by the "obvious" continuity principle, BF and BD would have two points in common, in violation of a principle that we read into the first postulate.

Çağdaş finished the presentations with I.21. Then I observed that somebody—it was the Epicureans, according to Proclus, but I had forgotten this—somebody had ridiculed Euclid for proving propositions like I.21 and especially I.20 (which it depends on), when they are obvious even to an ass, a donkey. This got some laughs, perhaps more so because I had given the Turkish word for donkey, *eşek*; the English might not have been in their vocabulary.

Of those registered students whom I had never seen, three—Rashad, Tuğba, and Nur—were in class finally. They claimed they knew what they were getting in for by registering for the course. Everybody else had presented propositions; so the newcomers became first in line for next Tuesday. After they agreed to this, other students were eager to sign up for propositions, so I wrote down their names too.

I ended the class by raising the question of whether we could now improve the proof of I.8, the SSS rule for congruence of triangles. Euclid's proof involves "applying" one triangle to another; I asked whether we could avoid this. (I had apparently taken up this issue in an essay I wrote at St John's. I dug up the essay this August when I visited my mother and went through those old things of mine that remained in her house.)

1.5. Tuesday, October 13

Today, in fifty minutes, we covered four propositions. Tuğba was first up, for I.22 (construct a triangle, given the sides). Or that's what I thought; she seemed to have prepared I.20

instead. Well, last class was the first one she actually attended; but Taner proved I.20 then. I asked Tuğba if she could prove I.22 anyway, but she preferred to present a proposition next time.

I thought Rashad volunteered to prove I.22. He came to the board and started writing down—Proposition I.23. This is the one he had signed up for. I said we couldn't do 23 before 22. He wasn't prepared to prove 22 (even though 23 uses it); so he sat back down.

Cihan volunteered to prove I.22, and did it. If the three given sides are A, B, and C, then as Euclid points out, we must have

$$A + B > C, \qquad A + C > B, \qquad B + C > A.$$

But how do these requirements come into the picture? I discussed this with Cihan, and we drew three pictures, with two non-intersecting circles each, showing what goes wrong when any of the three inequalities above is violated.

But Özge at least was brave enough to say, in effect, that she didn't get it. She came up to the board, and we discussed the matter some more, until she was satisfied.

Rashad then proved I.23: to construct a given angle on a given straight line and given point on it. One does this by constructing a triangle with the desired angle. I observed that I.22 hadn't spoken about **where** one could construct the triangle. I invited Rashad to show how to construct the desired triangle in the **place** where it was desired. He did this, easily.

Proposition I.24 was Nur's: If two triangles have two sides equal to two sides respectively, but the one included angle is greater than the other, then the side opposite the one is greater than the side opposite the other. In Heath's diagram

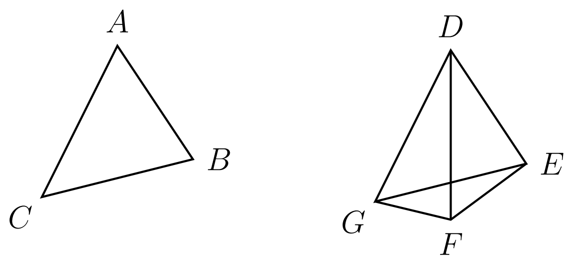

Figure 1.6. Euclid's I.24

(Fig. 1.6), Nur chose the point G merely to satisfy $DG = AC$. When I asked if there were any other condition, she said No. Eventually she just saw, or remembered, or saw in the notebook that she had set aside, that angles EDG and BAC should be equal.

Nur completed the proof as Euclid gives it. Then I asked: What if the point F happens to fall inside triangle DEG? I suppose I'm glad I hadn't consulted Heath's commentary on this proposition; Heath does discuss this other case, and he gives a simple proof—which I had overlooked.*

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*Editing in December, 2012, I am not sure what "simple proof" I meant. Probably it was the proof using I.21, which I did mention later. Heath also points out that we may assume $AC \geqslant AB$; but that in this case we should prove that the point F falls below EG. He refers to De Morgan's proposed lemma, that every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides. De Morgan reportedly proves this from his corollary to I.21, that among straight lines from a point to a give straight line, the perpendicular is the shortest, and they get longer as they move away from the perpendicular. As an alternative to De Morgan's argument, Heath suggests "the method employed by Pfleiderer, Lardner, and Todhunter," whereby, in the original diagram for I.24, we let DF (extended if necessary) meet EG at H (see the figure at the end of this note). Then $DHG > DEG$ by I.16, and $DEG \geqslant DGE$ by I.18, so $DHG > DGE$, and therefore $DG > DH$ by

Nur and others claimed that, in the picture I had drawn (as in Fig. 1.7), with F inside DEG (but $DF = DG$), it is obvious

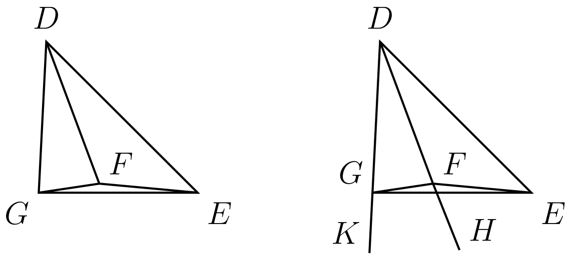
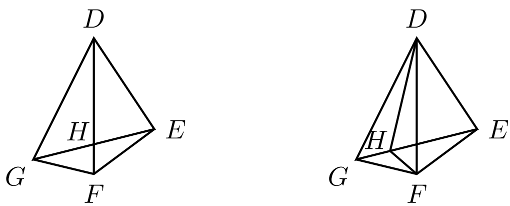


Figure 1.7. Alternative diagram for I.24

that EF is shorter than EG. Could they prove it? Well, they didn't think it needed proof. I claimed it did.

Somebody could have cited I.21, whereby $EF + FD < EG + GD$, which yields the claim. But nobody did. I suggested extending DG and DF, so that the **exterior** angles at the base of isosceles triangle DFG are equal. Thus the second part of I.5 could be used, in the same way that the first part is used for the case that Euclid does give. (I'm glad I hadn't consulted Heath, because I might not have noticed this argument if I had.)

I.19, so $DF > DH$, and F must fall below EG, as desired. Finally, Heath suggests a "modern alternative proof" of the proposition, all cases at once; this involves the bisector of angle FDG, meeting EG at H.



Checking Heath now at I.7, I learn that Proclus similarly proved the omitted case of that proposition. (I have the Proclus, but have not been reading him systematically; maybe I should.)

By this time, the class was almost over. I don't know if the students were happy that we covered so few propositions. In the remaining minutes, Cihan proved I.25 (the converse of I.24), which has a purely logical proof, I would say. There was no time to observe that I.6 (equal angles are subtended by equal sides) could have postponed till after I.18 (the greater side subtends the greater angle); then this with I.5 (equal sides subtend equal angles) would have allowed a purely logical proof of I.6.

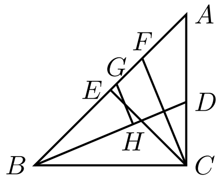
As I say, there was no time to discuss the alternative proof of I.6. Maybe next time, or not. I don't mind going slowly, if there are things to say; but I do want to get to mathematics that is more difficult in a conventional sense—such as the theory of proportion.

Perhaps next time I'll mention the Steiner–Lehmus Theorem without naming it—so the students can't just look up the proof, although that's what I did. Sam Kutler told us about this theorem when I was in his math tutorial, but we didn't discuss the proof. Conway's argument—which I read today—that there can be no **direct** proof is intriguing.

Note: The *Wikipedia* article on the Steiner–Lehmus Theorem provides a link to

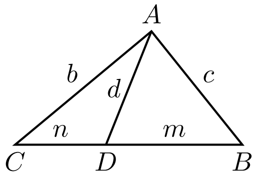
`http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus`

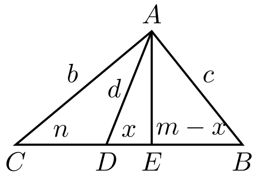
(accessed December 6, 2012), which discusses a number of proofs. Here is an adaptation of the first of these proofs (said to be taken from Coxeter's *Introduction to Geometry,* which cites a letter from H.G. Forder):



Assume triangle ABC is not isosceles. We may assume the angle at B is smaller than the angle at C [I.18]. The straight line BD bisects the angle at B, and CE bisects the angle at C [I.9]. We construct angle ECF to be equal to DBA [I.23], we let $BG = CF$ [I.3], and we let angle BGH be equal to angle BFC [I.23]. Then triangle BGH is equal in all respects to triangle CFE [I.26]. In particular, $BH = CE$. Now, $GH \parallel FC$ [I.28]. Also $BG < BF$, since $BF > FC$ [I.19]. Thus $BH < BD$. Therefore $CE < BD$. The contrapositive is that if $CE = BD$, then the triangle *is* isosceles. We have not used the Fifth Postulate.

An algebraic argument runs as follows. We first establish Stewart's Theorem:





For now, b, c, and d are simply lengths; in particular, they are positive elements of $\mathbb{R}$. We let a be the length of CB. But we conceive of n as a *vector* in $\mathbb{R}$ from C to D, and m as a vector from D to B, so that

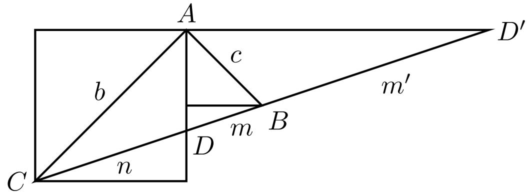
$$a = n + m.$$

We may allow D to fall to the left of C, in which case n is negative; if to the right of B, then m is negative. Similarly x is a vector from D to E, where E is the foot of the perpendicular dropped from A. Now we compute:

$$\begin{gathered}
d^2 - x^2 = b^2 - (n + x)^2, \\
d^2 = b^2 - n^2 - 2nx, \\
d^2 = c^2 - m^2 + 2mx, \\
ad^2 = b^2m + c^2n - mn^2 - nm^2,
\end{gathered}$$

$$a(d^2 + mn) = b^2 m + c^2 n.$$

In case the cevian AD is an angle bisector, we have $b/c = n/m$; this will be true, even if AD bisects an *external* angle at A, provided we then allow b/c to be negative (because n/m will be negative):



In this case then,

$$m = a - n = a - \frac{bm}{c}, \qquad m = \frac{ac}{b+c}, \qquad n = \frac{ab}{b+c},$$

so Stewart's Theorem yields

$$d^2 = bc\left(1 - \left(\frac{a}{b+c}\right)^2\right) = bc \cdot \frac{(b+c)^2 - a^2}{(b+c)^2} = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2}.$$

Here, if AD is the bisector of the *external* angle at A, then $b+c$ will be the *difference* between the lengths of AC and AB, and this is less than A [I.20]; but also $bc < 0$, so d^2 is still positive. In any case, $a+b+c$ is positive.

Suppose the (internal) bisector of the angle at B also has length d. Then

$$0 = \frac{a(c+a-b)}{(a+c)^2} - \frac{b(c+b-a)}{(b+c)^2}.$$

Here, if AD is the bisector of the *external* angle at A, then we must take b to be negative, while a and c are positive. In any case, the right-hand member here has $a-b$ as a factor. Indeed, the right-hand member is equal to the following.

$$\frac{c\bigl(a(b+c)^2 - b(a+c)^2\bigr) + (a-b)\bigl(a(b+c)^2 + b(a+c)^2\bigr)}{(a+c)^2(b+c)^2},$$

$$(a-b) \cdot \frac{c(c^2 - ab) + (a+b)(c^2 + ab) + 4abc}{(a+c)^2(b+c)^2},$$

$$(a-b)\cdot\frac{c^2(a+b+c)+ab(a+b-c)+4abc}{(a+c)^2(b+c)^2}.$$

In case $b>0$, the other factor is positive, so $a-b=0$, and the triangle ABC is isosceles.

John Conway (as quoted at the link above) argues that this is not a direct proof, since our computations alone do not yield $a=b$, but we need also that b is positive. If $b<0$, then we may have for example $a=c=1$, in which case

$$b^2+5b+2=0, \qquad b=\frac{-5\pm\sqrt{17}}{2}, \qquad b\approx -0.438$$

(the other root is too large in absolute value).

1.6. Friday, October 16

Today, Propositions I.26–33 were presented by Tolgay, Besmir, Jeremy, Seçil, Elif, Özge, Taner, and Yunus, respectively.

I had an evil thought today: that some students might be making long presentations in order to make sure we don't cover much material. It's probably false, but Ayşe has learned from students that they have tricks for slowing down her classes by getting her to talk about irrelevant things.

Tolgay proved I.26: the triangle-congruence theorem that we might call ASA and AAS. He got a bit confused at the board. I'm not sure how many of his classmates were paying close attention on a warm Friday afternoon in October. Did they see clearly where he needed to go, as he did not? At least Tolgay was learning: what you think you understand when you are by yourself may become strange when you are standing up in front of others.

I dragged things out after Tolgay wrote:

$$\angle AHB=\angle DFE;$$

$$\angle ACB = \angle DFE;$$
$$\angle AHB > \angle ACB,$$

which is impossible. (See Fig. 1.8.) I wanted him to spell out

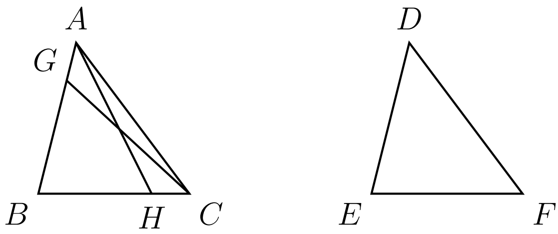


Figure 1.8. Euclid's I.26

the intermediate conclusion that $\angle AHB = \angle ACB$.

Talking to me once after class, Jeremy heard me say that I thought students should be able to express Euclid's enunciations in their own words, while being reasonably faithful to Euclid's style. In presenting I.28, Jeremy did write down the enunciation in his own words; and he remarked that he was doing so. But really, I didn't think he needed to take the time to do that in writing; like most of his classmates, he could have drawn a picture and just worked with that. But Jeremy does seem to aim at giving crisp presentations; so perhaps he sets a good example.

After Özge proved I.29 (parallel lines make equal alternate angles, etc.), we took a break. During the break, Cihan asked me if this was the first time we had used the Fifth Postulate, and if so, didn't that mean that everything before I.29 was true without the Fifth Postulate. Yes, I said, except that (as I had suggested at the time) I.16 raises a question for some commentators.

Ali wasn't sure that we had really needed Postulate 5 in I.29. Here, finally, I said we should talk about this in class. Ali seems to be one of the most attentive students; if he is confused at this point, others must be.

In class then, I had Ali raise his question. I reviewed a case of a "logical" theorem, such as I.19 ["In any triangle the greater angle is subtended by the greater side"], which follows immediately from I.5 [equal sides subtend equal angles] and I.18 [the greater side subtends the greater angle]. I asked whether I.29 could follow in such a way from its converse, I.27. Ali, at least, agreed that it couldn't. I couldn't resist mentioning Lobachevsky and his working out the consequences of the negation of I.29.

After Taner presented I.32 (an exterior angle is equal to the two opposite interior angles, and all interior angles are equal to two right angles), I asked him whether he had been familiar with the fact before. Yes, he said, but he had never proved it. I had been wondering just how familiar all these propositions seemed to the students. They confirmed for me my understanding that, on the university entrance exam, they just have to be able to compute numbers (perhaps angle measures in a geometrical figure).

I told Taner I was sorry he hadn't proved these propositions before. He said he was sorry too.

At some point I asked who was in the geometry class of one of my colleagues, Cem *Bey.* Cihan, Taner, and maybe some others said they were.

What are you doing? I asked.

Proofs! said Taner. He didn't know the English, he said, but they were studying things like the *ağır merkezi.*

Oh, the center of gravity, I said.

Yes, and the Nine Point Circle, said Cihan.

I wrote down the statement of the Steiner–Lehmus Theorem without naming it; but Cihan knew the name—from that Geometry class, apparently. Anyway, I need to talk to Cem *Bey.*

On other days, there have been only just enough desks in our classroom. If that had been true today, I was going to push them all to the edge. In fact, I told this to some students before class, and one of them asked, “Why?”

I didn’t say it was because of talk of Harkness Tables and so forth on the J-list. I just said everybody was a sort of teacher in class, and I didn’t want people sitting behind others and chit-chatting—as had happened a little bit before.

Well, there were a lot more desks in the room this time, and I didn’t feel like pushing them all around, but maybe I should have. Rashad sat in back playing with his cell phone. Taner and Seçil, conferring over a desk, said they were working on the mathematics; I said we should all talk about it together, but nothing really came of this.

I was going to say here that my class was not quite St John’s, but then I remembered a Johnny classmate who used to sit at the seminar table reading science fiction novels in her lap.

1.7. Thursday, October 22

On Tuesday, October 20, I did arrange the classroom desks in a sort of semicircle. When some students walked in late and tried to go behind, their classmates indicated that they should find a seat in the circle. Students moved their desks to accommodate the newcomers.

Propositions I.34–39 were presented by Ahmet, Mürsel, Tuğba, Melis, Nur, and Rashad.

I.35 ("parallelograms which are on the same base and in the same parallels are equal to one another") is supposedly the place where the meaning of equality changes. Indeed, Mürsel wrote the statement on the board as I have quoted it, but after "equal" he inserted a parenthetical comment: "(equal in area)." After his presentation, I asked what "area" was, claiming that we had no notion of area as a number (even with units, as 5 cm^2). Euclid just says the two parallelograms are equal, and I don't see any difference in meaning from I.4, where the statement is:

> If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

If by saying the two triangles were equal here, Euclid had meant they were congruent, then he would not have bothered to mention the further equalities of sides and of angles.

It could have been after I.36, but I think it was after I.38 when Jeremy asked: why didn't Euclid just use the method of "application," as in I.4 and I.8? After all, I.36 is the variant of I.35 where the parallelograms are on **equal** bases. Then I.37 is like I.35, but is about **triangles** on the same base; and I.38 is about triangles on equal bases.

So in proving I.36 and I.38, why didn't Euclid just say, Apply one figure to the other so that they have the same base? I just suggested that Euclid preferred to avoid using the method of application whenever possible. That's perhaps an inadequate answer, since he could have avoided the method in proving I.8, but didn't.

It occurs to me only now that a better answer to Jeremy's question would be that, after applying one figure to the other, one can no longer be sure without proof that the two figures are in the same parallels.

Jeremy addressed his question to me, calling me "Sir." If it happens again, I'll ask him at least to use the Turkish *Hocam* ("My teacher"; but even bus drivers are *Hocam* at our university). Really Jeremy should have addressed the whole class with his question; but I don't know what more I can do than I have already done to raise interest in general discussion.

After their presentations, students **are** uttering a *pro forma* "Any questions?" to their classmates.

It is interesting to see the different styles of students at the board. Some make their arguments almost entirely out loud, and I ask them to write down some of the details. Others try to write down everything, and I suggest that they can leave some things out (especially the general enunciation).

I hope I'm not micromanaging. From freshman mathematics at St John's College, I do recall a time when one student was drawing a straight line from A to B, and Mr Kutler suggested it would be better drawn from B to A.

Back in Ankara, at the end of class, I took volunteers for the remaining propositions in Book I. Already Çağdaş had been lined up for I.40; but since that proposition seems not to be a Euclid original, I asked Çağdaş if he would do I.41 instead. He agreed, mentioning his awareness that it was Heiberg who had declared I.40 an interpolation.

After Book I, I plan to lecture on the theory of proportion. There are two challenges to reading Euclid: the mathematics, and Euclid's style of doing mathematics. For proportion at least, I shall try to mitigate the latter challenge through the use of symbolism. I am aware that written symbolism is subject to the same criticism that Socrates levelled at writing in general: It allows the brain to get lazy.

In any case, I want to challenge the students with Apollonius as soon as possible.

I talked some time on Tuesday morning with Cem *Bey,* who seems quite pleased to learn about my course. He asked if I would cover non-Euclidean geometry in my own course, since, for himself, the discovery of non-Euclidean geometry was more important than landing on the moon. That sounded naive to me: of course the geometry was more important! In any case, I said I would do non-Euclidean geometry in the spring semester, if I was assigned the modern math history course.

Cem *Bey* confirmed that there was no Turkish translation of Euclid; there had just been an Ottoman translation of a geometry text used at Sandhurst. (Cem *Bey* is, among other things, a scholar of Ottoman mathematics, and he detests the Turkish language reforms, which have deprived him of the expressiveness of Ottoman Turkish.)

Regarding Euclid, Cem *Bey* mentioned that we did have a colleague whose mother tongue was Greek; but I already knew this. (She was born in Rhodes, but took Turkish citizenship some time after coming to Ankara to study; she now needs a visa to visit her family back in Rhodes.)

Cem *Bey* was impressed to learn that there were "still" colleges like St John's in the West. Though he considered German almost as his mother tongue, he regretted not having been able to read Kant.

I don't really need more projects, but now I want to translate Euclid into Turkish. At least Turkish has a better way than English does for expressing Euclid's passive third-person perfect imperatives.

1.8. Saturday, October 24

My next installment is fairly rushed; but as usual I want to keep a record while the memory is fresh.

On Friday, October 24, we finished Book I. I don't know how well the students appreciated the **poetry** of Euclid, but I think they enjoy the class reasonably well. At least, I never have any trouble getting volunteers for presenting propositions.

I have wondered if it is tedious for some students to see their classmates work out laborious proofs of "obvious" facts. Even if the students accept that proofs are necessary, maybe they get frustrated to see a classmate struggle with what should be an "easy" proof. But I don't have any real evidence of this, and anyway the proofs are getting harder now.

Before class, Ali asked if we would sit in a semicircle again. He didn't mind sitting in the front row, he said, but maybe other students did. I decided just to leave the chairs as they were, in a rectangular array. I didn't notice any chatting in the back this time.

Moreover, I had a couple of guests, one of whom I knew from the Nesin Mathematics Village in the summer of 2008. She's only now a first-year student in our department. I don't know what she had heard about my class, but she came and listened and took notes. She had a male companion, but I don't know if he was equally interested. Indeed, my spouse noticed a guest in her linear algebra that day: he seemed to be the boyfriend of one of the students.

Ali also asked me whether we had Euclid's text only because of the Arabs. I said there were some texts of which this was the case, but I didn't think it was so for Euclid.

Ali also asked why we were skipping I.40. He didn't understand how Heiberg could decide that it was an interpolation.

I didn't know how, but recalled something about a papyrus fragment mentioned by Heath. Anyway, Ali agreed that I.40 was not particularly surprising or important.

After Çağdaş has proved I.41, when Ali started proving I.42 (construction of a parallelogram in a given angle equal to a given triangle), he asked if his triangle was too small. Nobody complained, so he continued. But he stood right in front of his picture, facing and talking to the blackboard. I suggested that many people couldn't see, but he just said "I already asked if my picture was too small."

Tolga made his first presentation with I.43 (the complements of parallelograms about the diameter of a parallelogram are equal). The previous class had been the first that he attended. It wasn't too clear that he understood what he was proving. He may have been confused by Euclid's convention of writing EG and HF to designate **parallelograms.** He seemed surprised when I drew attention to the convention, although he had supposedly finished his proof. I tried to get him to shade the two equal parallelograms EG and HF, but he didn't understand the request until somebody else explained in Turkish.

I.44 (construction of a parallelogram in a given angle, on a given side, equal to a given triangle) was pleasant. It's nice how Euclid proves that HB and FE must actually meet when produced. But Özge was vague about what the point L was, so I inquired about this. It turns out that here, as elsewhere, Euclid does not describe the construction of the figure, but talks about a figure that has already been constructed. Özge drew KL parallel **and equal to** FH. Euclid just draws it parallel, and L is determined because this is where it meets HA extended; but Euclid names KL before he has even referred to the extension of HA. I seem to recall being a bit disconcerted, as a student, by this habit; but I don't recall considering the reason for the habit. On Friday, I went to the board to ask why HA and (the line that ends up being) KL can't be parallel. I thought of using the Fifth Postulate again, but the students told me that if HA and KL were parallel, then the intersecting lines FE and FG would be parallel.

Ahmet proved I.45 (construction of a parallelogram equal to a given rectilinear figure). He still wanted to say that the two figures had equal **areas,** not that the figures were themselves equal. But it was break time, so I postponed my complaint till after that.

Then I drew three lines on the board: one straight, two curved. I asked: Do they have lengths? I asked for a show of hands; most people said the lines **did** have lengths; a couple said No.

I asked, If you think this curved line has a length, what is it?

Jeremy said, We need a unit.

I drew a unit. He said, We need a smaller unit.

I acknowledged that with calculus one could **define** the length of a curved line; but we couldn't do it with the tools at hand. I wanted to argue that it was meaningless to abstract a "length" from a line unless you could compare two lines (for example) and say they were equal; then you might define "length" as "that by virtue of which two equal lines are equal." But Euclid gives us no way to compare curved and straight lines; so it is meaningless to talk about the length of a curved line. Rashad wanted to be able to pull a curved line straight; I observed that we had no such postulate.

Jeremy argued that we should still allow a concept of length, for the sake of philosophy, or something like that. I said we should avoid talking nonsense, to keep philosophy from getting a bad name.

With plane figures, the matter is different. Euclid does give us the means to compare them. Now we know that triangles can be "equal" to parallelograms and other figures. OK then, what makes them equal is their "areas," if you like; but we are still far from having "area" as a number.

Today we may approximate areas by of figures by dividing them into little squares. Euclid turns rectilinear figures into parallelograms by I.45; but the parallelograms need not even be rectangles. Here I uttered my complaint that we today were obsessed with **right** angles: that every angle on our campus, in fact, was right.

Back during the break, Çağdaş had looked at my Green Lion edition [23] of Euclid and wondered about the claim on the back that Euclid was the most celebrated mathematician of all time. Did I agree with that? he asked me. He seemed to think that *somebody* was going to write some such compendium as Euclid's; Euclid just happened to be the one who did it. Maybe I didn't understand his idea.

After Besmir constructed a square in I.46, Rashad gave a careful proof of I.47. Jeremy asked why we call it Pythagorean, if Euclid proved it. Rashad said that Euclid's theorem was different: Pythagoras's theorem was

$$a^2 + b^2 = c^2.$$

I told Ayşe about this later. I thought Rashad meant that Pythagoras was interested in identifying "Pythagorean triples," like $(3, 4, 5)$ or $(7, 12, 13)$; but Ayşe suggested that possibly Rashad didn't actually see the connection between a geometrical square and the square of a number.

Time was running out, but I thought we should fit in I.48 to complete Book I, and Seçil was ready. It's a nice proof: In proving a converse, Euclid often uses the method of contradiction; he could do the same for I.48, but he avoids this and gives a direct proof.

Next time, a few propositions from Book II. I wonder what the volunteers will make of them.

1.9. Wednesday, October 28

On Tuesday, October 27, I had five students lined up for the first five propositions of Book II. Jeremy was number 1; but Ali told me that Jeremy couldn't come: something about seeing the police concerning his residence permit. Jeremy hadn't asked Ali to present II.1 in his place, but I asked Ali if he would do it anyway, and he agreed.

Class hadn't started yet, so Ali went on to ask me whether Euclid does define "area" somewhere. It seems these kids are obsessed with assigning numbers to geometrical figures. Without numbers, it's not math! That's what the system seems to teach them.

I mentioned that Euclid was surely aware of the desirability of assigning numerical areas to plots of land; it just wasn't his interest in the *Elements.* The *Elements* ends with the construction of the five Platonic solids, I said; Ali seemed to find this exciting.

Five minutes into class, there were still only about five students. One of them speculated that the others had gone home for the holiday: Thursday is Republic Day. Friday is not an official holiday, but I cancelled class anyway so that Ayşe and I can take a long weekend down by the Mediterranean, and the students can have a break too. But I didn't mean for them to take the whole week off.

Some time during Ali's presentation, several other students walked in.

Ali used the notation (A, B) for the rectangle contained by A and B. When Mehmet presented II.2, he used $\mathrm{R}(A, B)$ for this rectangle, and $\mathrm{S}(A)$ for the square on A. I invited him to write a modern algebraic formulation of the proposition, and he did.

Tolga's presentation of II.3 was confused, but it was Ali and somebody else (Cihan?) who got him to straighten things out, not I.

Yunus took a long time with II.4, though in the end he seemed to be able to recover a rigorous demonstration in the manner of Euclid. When I asked him, he admitted that he didn't see the point of proving the proposition, since it is obvious that $(x + y)^2 = x^2 + 2xy + y^2$. I suggested that this equation is just symbols, but the geometry is the "real thing." I think I mentioned that Descartes thought the Greeks must have had some sort of algebra. I don't know, myself. When I worked through Book I of Apollonius in 2008, I had the impression that he did **not** have the sort of algebraic point of view that I could adopt.

I asked Özge to postpone her presentation of II.5 till next week. I used the remaining few minutes to give a preview of proportion. I stated VI.1 in words and also in the form

$$ABC : ACD :: BC : CD.$$

But what does this **mean?** I asked. I wrote it in the form that the students would expect:

$$\frac{\text{area}(ABC)}{\text{area}(ACD)} = \frac{|BC|}{|CD|},$$

but argued that we didn't know what this meant. Actually, Ali and others might argue that they do know what this means, with calculus.

A thought about Book II: Heath suggests that Euclid proves the first ten propositions independently because he is mainly interested in establishing a **method.** He could derive II.2 from II.1, for example, but that's not the point. In Book I though, it **is** the point, or **a** point: I mean, Euclid's bisection of an angle in Proposition 9 is not the most efficient; but it relies on Proposition 1, and perhaps for this reason Euclid prefers the construction he gives to an independent construction. It's as in a joke:*

> How does a mathematician boil water? By filling the kettle, putting it on the stove, and turning on the flame.
>
> What if the flame is already on? Then the mathematician turns it off, thus reducing the problem to the previous problem.

1.10. Friday, November 6

Class on Tuesday, November 3, was a lecture by me. Özge was going to present II.5, but she wasn't there on time, so I just jumped ahead and presented II.10. In algebraic formulation, this is

$$(2x + y)^2 + y^2 = 2x^2 + 2(x + y)^2,$$

which can be rearranged to form the equation

$$(2x + y)^2 - 2(x + y)^2 = y^2 - 2x^2.$$

*The joke is based on a section of Smullyan's book *What is the Name of This Book?* [47].

In particular, if (a, b) is a solution to

$$2x^2 - y^2 = 1,$$

then $(a + b, 2a + b)$ is a solution to

$$y^2 - 2x^2 = 1.$$

So we can generate a sequence

$$(1, 1), (2, 3), (5, 7), (12, 17), (29, 41), \dots, (a_n, b_n), \dots$$

where b_n/a_n tends towards the square root of 2. This is what we "know" today; but what does it **mean?** What is "the square root of 2"?

I proved VI.2, having assumed VI.1 without proving it or even defining proportions exactly. Then I went back to discuss the definitions in Book V of having a ratio and having the same ratio. Then I proved VI.1.

I saw a lot of sleepy faces as I stood at the blackboard. This reminded me that it had been good when the **students** were doing the presenting at the blackboard.

Well, today I give an exam on Book I, which is why I didn't want to trouble them to give presentations on Tuesday. The exam problems are [see also §A.1]:

1. To find the error in a proof that all triangles are isosceles. (I don't know if some students will have seen this in some popular book.)
2. To translate some Greek words (like *θεώρημα* and *πολύγωνον*) into English.
3. To write down the Greek alphabet.
4. To give a proof of I.6, analyzed into the six parts described by Proclus (enunciation, exposition, specification, construction, proof, and conclusion). A confusing point here is that Euclid's proof is by contradiction,

so the "construction" step is based on a hypothesis that turns out to be false. So what part does this false hypothesis lie in? I don't know whether Proclus contemplated this question. One doesn't really **need** the false hypothesis though, one can just construct the point D, which in the end turns out to be the same as A.

5. To prove I.8 ("SSS") without using Euclid's method of application. I had invited the students once or twice to consider this problem.
6. Something new: In triangle ABC, suppose BC is bisected at D, and straight line AD is drawn. Assuming AB is greater than AC, prove that angle BAD is less than DAC. It's possible that few will get this, but I want to find out.

Meanwhile, at the end of class on Tuesday, I lined up volunteers for next Tuesday to present some propositions about circles from Book III: 3, 20, 21, 22, and 31. I chose these because they seem to be needed for Apollonius, and I am keen to get to him. ("Had we but world enough and time," we would just read all of Euclid. "But at my back I always hear time's winged chariot hurrying near.") After Book III, I'll get students to present from Book V, so they can deal with proportionality themselves.

1.11. Friday, November 13

Last Friday, November 6, I gave the students in "math history" class a written exam. I was depressed afterwards, because I had the impression that the students had not done well. I feared that my own enthusiasm had blinded me to the difficulties that the students must have with Euclid.

After I read the exams on Saturday morning, I felt a lot better. I saw that I had not been wrong to put a "hard" problem on the exam. Several students found a better proof than the one I had thought of.

On another problem, a student introduced a novel method of proof. Maybe he didn't clearly see that he was doing this, but: If you are given that angle ABC is greater than angle DEF, doesn't that mean that there is **some** straight line AD drawn inside angle ABC so that angle ABD is equal to angle DEF? It would seem so, except that Euclid insists on being able to construct AD.

Unfortunately most students had not taken seriously my demand that they learn the Greek alphabet. But some had, including one, Taner, who had once complained [p. 48] about doing math in English rather than Turkish.

When I met with my set theory study group on Saturday, our classroom had the Greek alphabet on the board, with a few mistakes. I recognized the hand as that of Tolgay—who had not made those mistakes on the exam.

I put a photo of that blackboard on the course webpage, along with solutions and commentary on the exam.

On Tuesday, November 10, Özge presented II.5. I was glad I had asked her to do this, because this provided an opportunity for discussing problems like: Given a straight line AB and a square C, how can we find point D on it so that the rectangle contained by AD and DB is equal to C? I worked this out **analytically,** that is, by assuming that we already have D and working backwards. I gave the students the exercise of doing the same thing, only with D on AB **extended.**

I also did II.11—to divide a straight line in extreme and mean ratio (though the terminology is not available at this point yet)—again in the analytic style, though not in Descartes's algebraic style. It seems to me quite plausible now that, as Mr Thomas reported on the J-list, Newton also thought about his work in the ancient style, visually, rather than by symbol manipulation. Actually, Mr Thomas wrote:

> One of the things that I **knew** coming out of St John's was that Newton derived his results "analytically" and then cast them into "synthetic" form *à la* Euclid. Cohen [40] tells us, however, that there is "no shred" of evidence that this was so. And Newton apparently never threw anything away, so the absence of evidence is telling.

So I wonder if "analytic" and "synthetic" are the right words here. But I haven't got Newton with me yet.

Meanwhile, today, November 13, half the students didn't show up. I was told that they had an exam in another class right after mine, and they wanted to study. Their teacher was a friend we went out with last night, actually. Anyway, students presented a few propositions about circles from Book III that I had asked for, and then we moved on to Book V. But I haven't time to say much about that yet. At the beginning of class, I did discuss a common English error [at least by native Turkish speakers]: to write

Let AB is the given straight line

rather than

Let AB be the given straight line.

I talked about the Turkish subjunctive and imperative verbs and noted the English periphrastic equivalents.

1.12. Wednesday, November 18

On November 13, Elif began with III.3, namely: In a circle, a diameter bisects a chord that is not a diameter if and only if the two are at right angles. (I note by the way that Euclid does not seem to use the word "chord.")

In Book III, I tried to select for presentation only those propositions that would be needed for Apollonius. To do this, I relied on the editors of Euclid and Apollonius. Proposition III.3 really relies on III.1: To find the center of a given circle. But the Green Lion edition doesn't indicate as much. I didn't notice this until Elif presented III.3. I asked her if she could prove III.1, but she couldn't. I asked if somebody else could do it, and Cihan said he could. He did it too, by taking the intersection of the perpendicular bisectors of two chords with a common endpoint. Euclid doesn't do this: he take the midpoint of the perpendicular bisector of one chord.

Is III.1 really required for III.3? The former is a "problem," the latter a "theorem." The latter simply needs to know that the center of a circle exists; but it does exist, by definition of a circle. Possibly this is why the Green Lion editors, or Heath before them, left off a reference to III.1.

But Euclid does not seem to rely on the existence of something unless he can actually construct it. Later I shall mention an exception to this, in Book V. Meanwhile, I think that, in Euclid's own terms, III.3 relies on III.1.

Jeremy was supposed to do III.20, but he was missing, so I did it. The theorem was no surprise to the students: the angle at the center is double the angle at the circumference. Cihan raised the question of what happens when the "angle at the circumference" is considered as drawn to the smaller half of the circumference.

Nur proved III.21: angles in the same segment are equal. I asked her when she had first learned this. Before high school, she said.

Tuğba presented III.22: the opposite angles in a quadrilateral inscribed in a circle are equal to two right angles.

Taner presented III.31, but said he was confused about something. I discounted this, until I realized that there really was something strange. The proposition is mainly that the angle **in** a semicircle is right; that the angle in a segment that is greater than a semicircle is less than a right angle; and less, greater. Taner had no problem with this.

But then Euclid says the angle **of** a segment that is greater than a semicircle is greater than a right angle. He's talking about a **curvilinear** angle. The same sort of angle is mentioned in III.16, but we had skipped this: " . . . further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle."

In *Euclid: The Creation of Mathematics* [9, p. 83], by Benno Artmann, I read the claim that III.16 shows that Euclid was aware of "non-Archimedean" orderings. That's a strong claim. In the language of Book V, the angle between a tangent and a circle does not **have a ratio** with any rectilineal angle, since no multiple of the former can exceed the latter. But Euclid does not seem to remark on magnitudes that do not have a ratio. Throughout Book V, there often needs to be an assumption that certain magnitudes have a ratio; Euclid does not mention this assumption. One might wonder whether different people compiled Book V and the earlier books.

In class we moved on to Book V ourselves. Cihan presented V.1; Yunus, V.2. But there was not much to present. Unfortunately, again I don't recall clearly what happened: it was five days ago.

Actually I do recall that Yunus misstated his proposition first, so we corrected it. Euclid states V.2 in terms of six magnitudes, a first through a sixth. If he was going to use algebraic notation, I thought Yunus might call these magnitudes A_1, A_2, . . . , A_6. But he didn't.

I must have got up to write my own algebraic formulation of these propositions:

$$nA_1 + nA_2 + \cdots + nA_m = n(A_1 + A_2 + \cdots + A_m),$$
$$n_1A + n_2A + \cdots + n_mA = (n_1 + n_2 + \cdots + n_m)A.$$

Here capital letters are magnitudes, and minuscules are multipliers.

It does appear that Euclid himself treats multipliers in isolation in one place, VII.39: "To find the number which is the least that will have given parts." He means for example to find a number that will have a third part, a fourth part, and a seventh part. In this case we must find the least number A for which there are numbers B, C, and D such that $A = 3B = 4C = 7D$.

But in the general situation, Euclid says, in effect, let E, F, and G be the given parts. But then we can't just apply Proposition VII.36 to this, taking the least number measured by E, F, and G. No, first we have to take **numbers** H, K, and L that are "called by the same name as" E, F, and G. So in my example, E, F, and G are not three, four, and seven; they are third, fourth, and seventh. But third, fourth, and seventh what? I don't know. Proposition VII.39 is the last in Book VII, so maybe it was added later. Heath doesn't suggest this however; nor does he remark on its strangeness. He says only that VII.39 is "practically a restatement" of VII.36. If so, then we really should inquire why Euclid makes the restatement.

Back to my class, and my equations above. I suggested to the class that magnitudes and their multipliers were like vectors and scalars in linear algebra. In particular, the two equations above are certainly not "practically the same." This point comes out to us if we write (as I did in class) V.3 as

$$k(mA) = (km)A.$$

Multipliers can multiply each other; magnitudes as such cannot.

I hadn't actually assigned V.3. Tolga had volunteered for V.4, but was not present. I presented it. It is the first proposition about proportions: Symbolically,

$$\text{if } A : B :: C : D, \text{ then } kA : mB :: kC : mD.$$

I should have assigned V.3, since V.4 uses it.

Mehmet was supposed to present V.7, but we were out of time. That's good, because he and I got to talk about his proposition before he presented it. He asked me after class whether V.7 wasn't simply

$$\text{if } a = b, \text{ then } a/c = b/c,$$

and if so, isn't it obvious? I said:

1. If you think it is obvious, you can say so when you present it.
2. Do not write fractions like a/c. Today we think of a fraction as a single thing, a number. But in Euclid we have no justification for thinking this way. We can think the way we do only because people like Euclid have done the groundwork.
3. Do not use the equals sign between ratios. Equality is a "common notion." If $A = B$ and $B = C$, then we know $A = C$. We don't know this with ratios, but must prove it; in fact it is V.11.

Unlike, say, "less than," proportion is not a relation between two things; it is a relation between **four** things. Such relations are almost unheard of, unless we want to take an expression like "These are my parents, and this is my sister" as signifying a single relation between four people. Moreover, proportion is something we **define.** So we cannot just "intuit" its properties; we have to prove them.

That's roughly what I said to Mehmet, except I didn't use the example of familial relations. I recall talking more, even bringing up what I learned from Mr Thomas, that there's no evidence that Newton did **not** think about mathematics the way the Ancients did. Mehmet is majoring in physics as well as mathematics. I don't know if he found it attractive to learn that, if I did get to teach the second semester of this course, I wanted to read Newton.

1.13. Friday, November 20

On Tuesday, November 17, Mehmet started math history class by presenting Euclid's Proposition V.7. He stated it as

$$\text{If } A = B \text{, then } A : C :: B : C \text{ and } C : A :: C : B$$

and then he declared that it was obvious. As I noted in my last entry, such a declaration is what I had suggested (if he thought it correct).

I said maybe the proposition is obvious, **after** one observes that

$$A : B :: C : D \text{ if and only if } C : D :: A : B$$

(the "::" relation is symmetric). Mehmet said, correctly I think, that all one really needs to observe is that

$$A : C :: A : C$$

(the "::" relation is reflexive). But then one needs a "substitution principle": if two things are equal, then one of them can be substituted for another in any mathematical statement.

Now, Euclid does not have a proposition to the effect that $A : C :: A : C$. Would he take such a proposition as obvious, or as pointless?

The Common Notions include 1, that things equal to the same are equal to each other. If we want to express this symbolically, we might write

$$\text{If } A = C \text{ and } B = C\text{, then } A = B.$$

But then we should also observe that

$$\text{If } A = B\text{, then } B = A,$$

so that, if we should find that $A = C$ and $C = B$, then $A = B$. However, there is no express Common Notion to the effect that, if a first thing is equal to a second thing, then the second thing is equal to the first. This is just tacitly understood. One does need it in modern mathematical arguments.

"A thing is equal to itself" is not a Common Notion either. It may be true, but, although devotees of Ayn Rand may worship the equation "$A = A$," I can't think of an occasion where such an equation is used in mathematics. This is why I suggest that it might be pointless for Euclid to prove $A : C :: A : C$.

Again, from $A : C :: A : C$, one could derive V.7; but again, this would be by a "principle of substitution," and such a principle could not very well be stated in Euclidean terms. Euclid is not in the business of manipulating formal expressions in some artificial "language." The best way for him to prove V.7 is probably just as he does it. In particular, V.7 is **not** obvious.

But I can subject my students to only so much of this speculation. Back in class, we moved on to V.8. Besmir was supposed to do it; but he thought he was supposed to prove III.8.

Other presenters in class knew we were in Book V; so the mistake must have been Besmir's.

I presented V.8 myself. It brought to light yet another difficulty with Book V. The claim is this, symbolically:

$$\text{If } A > B, \text{ then } A : C > B : C \text{ and } C : B > C : A.$$

Assume $A > B$. We want multiples of the various magnitudes so that

$$kA > mC, \text{ but } kB < mC;$$

we might write this as

$$kA > mC > kB.$$

To achieve this, we should make the gap between kA and kB greater than C. But

$$kA - kB = k(A - B),$$

by V.5. So we just take k large enough that

$$k(A - B) > C.$$

To do this, we assume that $A - B$ and C **have a ratio** in the sense of Definition V.4. More on this presently. Meanwhile, accepting this, we let mC be the first multiple of C that exceeds kB. Then $kA > mC > kB$, as desired.

In Heath's translation, Proposition V.8 is:

> Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

There is an implicit assumption: each of the three magnitudes mentioned does have a ratio to each of the rest. That's fine. But there's another assumption, which the proof requires: If two **different** magnitudes have a ratio to a third ratio, then so does their difference. This means, for example, we can't take the sum of a square A and a straight line B and call the result C: for then we should have, presumably, $C > A$, although $C : A :: A : A$.

Ali presented V.9; but he just said it followed immediately from V.8, being its contrapositive. I noted that it was the contrapositive, provided one noted that, of two unequal magnitudes, exactly one is greater than the other.

Ahmet presented V.11:

If $A : B :: C : D$ and $C : D :: E : F$, then $A : B :: E : F$

(the "::" relation is transitive). He just said it followed from the transitivity of implication: If A implies B and B implies C, then A implies C. He admitted it was slightly more complicated, because of the quantifiers: we assume

(1) for all k and m, if $kA > mB$, then $kC > mD$,

(2) for all k and m, if $kC > mD$, then $kE > mF$.

We want to conclude

(3) for all k and m, if $kA > mB$, then $kE > mF$.

Actually, using formal logic might obscure the point.

Çağdaş presented V.12: If

$$A_1 : B_1 :: A_2 : B_2 :: \cdots :: A_n : B_n,$$

then

$$A_k : B_k :: A_1 + A_2 + \cdots + A_n : B_1 + B_2 + \cdots + B_n.$$

I think it was here, and I think it was Ali who pointed out the assumption that each of these magnitudes should have a ratio to the others; otherwise we might be adding squares and straight lines, which would lead to problems as discussed above with V.8.

1.14. Friday, November 20

Mr Gorham asked on the J-list:

> Isn't reciprocity included in the notion of "equal to"? Maybe I'm thinking of it linguistically instead of mathematically but it seems to me there's no need to spell out that if $A = B$ then $B = A$ because "=" contains within it the idea of a two way street.

What do you mean by "included in"? When Euclid writes "Things equal to the same are equal to one another," well, indeed that reciprocal pronoun "one another"—I guess one would call it reciprocal, or something like that—that "one another" suggests the meaning, "A is equal to B, and B is equal to A."

But we modern mathematicians recognize that equality has three distinct properties:

Reflexivity $A = A$;
Symmetry if $A = B$ then $B = A$;
Transitivity if $A = B$ and $B = C$ then $A = C$.

It is of some interest that Euclid (or somebody writing under that name) distinguished only the last (or some formulation of the last) as a Common Notion (again, unless you want to read symmetry also into that Common Notion). However, the relation of "less than or equal to" is reflexive and transitive, but not symmetric. Other such examples show that no two of the properties imply the third.

Eva Brann's friend Barry Mazur has an article on his homepage* called "When is one thing equal to some other thing?" It's been a while since I read it, but the theme (as I recall) is the mystery about what equality is in mathematics.

1.15. Saturday, November 21

In the class of Friday, November 20, it was interesting to see the different styles of different students in presenting propositions from Book V.

Rashad began with V.14, using modern symbolism. He was a bit late, and before his arrival, I had ranted a bit about the irresponsibility of agreeing to present a proposition and then not showing up. Then Rashad entered in a rush.

Melis continued with V.15, following Euclid's style exactly. The enunciation is, "Parts have the same ratio as the same multiples of them taken in corresponding order." For convenience, I would write:

$$A : B :: kA : kB.$$

Melis just wrote out the words, and gave the proof as Euclid does, with a diagram like Euclid's, for the case where (in my notation above) $k = 3$. Such "proof by example" is perhaps considered short of rigorous today; at least, it's out of style. But what really is the problem with it?

I asked Melis, "What if there were seventeen of the part in the whole, instead of three?"

She said, "The proof would be the same." She's right.

Seçil presented V.16, which symbolically is

*I can't find the article at `http://www.math.harvard.edu/~mazur/` anymore, but it seems to have been published as [38].

$$\text{If } A : B :: C : D, \text{ then } A : C :: B : D.$$

She also followed Euclid closely, but I had the feeling that this was because she did not comprehend the proof very well. Actually she confused some letters, but had a bit of trouble correcting them when the mistake was pointed out. Well, I know one's brain can stop working well when one is standing at the blackboard; it had happened to me earlier in the day in our departmental algebra seminar.

Mürsel was next with V.17: "If magnitudes be proportional *componendo,* they will also be proportional *separando.*" But he didn't write out the words, and I don't think many of the students are using Heath's translation with those Latin expressions. Mürsel just gave a symbolic statement and proof.

Talha, volunteer for V.18, was missing. Actually he hadn't volunteered: I **assigned** almost everybody a proposition from Book V. Talha started attending class late, and he has never presented a proposition.

I presented V.18 myself, noting what seems to be a first for us in Euclid: an assertion of existence without construction. I mean, Euclid says that if CD is **not** to DF as AB is to BE, then CD must be to **some** DG as AB is to BE. Well, this seems to be a new postulate. We are in no position yet to **construct** a magnitude that has a given ratio to another.

In the book [9, Ch. 14, p. 134] that I mentioned another day, Benno Artmann passes on a claim that some propositions in Book V are copied verbatim from Eudoxus, since nobody wanted to change the words of the master. Maybe the proof of V.18 is evidence for this. [In fact Artmann was talking about V.8.]

Tolgay presented V.20, and Özge, V.22. Really, V.20 appears to be just a lemma for use in proving V.22, which is

If $A : B :: D : E$ and $B : C :: E : F$, then $A : C :: D : F$ *ex aequali.*

And that was all it seemed necessary to do from Book V. [This was wrong; I turned out to want V.23 for the final exam. I just gave it to the students then.]

I had done VI.1 and 2 on an earlier day; now Besmir did VI.4 (equiangular triangles have proportional sides). I asked how he knew that BA and ED met **beyond** A and D. The answer seemed to be that, if ED met BA between A and B, then ED would cross AC; but these two lines are parallel.

Elif presented VI.6: triangles with one equal angle, and the sides about it proportional, are equiangular. Euclid's is another peculiar proof, like that of I.48, where along one leg of a triangle, a new triangle is constructed that turns out to be congruent to the first. If the new triangle were constructed on the same side as the first, then it would coincide with the first; but Euclid wouldn't like this, so he would assume (by way of contradiction) that the triangles were **not** congruent. Thus the fact that a straight line has two sides allows Euclid sometimes to avoid proofs by contradiction.

Seçil was scheduled for V.8, but our time was almost up, and she was happy enough to postpone her presentation till Tuesday. Tolgay was scheduled for V.11, but he had already left, apparently to collect his thoughts before an exam immediately after my class. I took volunteers for the remaining propositions in Book VI.

1.16. Friday, December 11

I sent my last report on my "math history" class almost three weeks ago, on the class of Friday, November 20. Since then, there have been only three classes, one hour each. Friday, November 27, was the Feast of the Sacrifice. I have no classes on Thursday or Monday, so I got a five-day weekend; but I spent it at my desk at home, working on various projects. There was no real external compulsion to do this work, just my inner drive.

One (but only one) of my projects was preparing to give a talk the following weekend in Istanbul. There wasn't really much to do, since I could more or less repeat the talk I had given in France in the summer; but I made a lot of adjustments. The occasion was a 60th-birthday conference. I had met Oleg Belegradek when I was a student at Maryland. Then he was still working in Siberia; now we have both ended up in Turkey. For the birthday event, Oleg's former Kemerovo colleague, Boris Zil'ber, came from Oxford where he now works.

I cancelled math history class on Friday, December 4, to take an afternoon bus to Istanbul. It was a bad time of day to go to Istanbul; evening traffic held us up for an hour. The driver said the delay the previous evening had been two hours.

At the party on Saturday evening, somebody gave Oleg a present: *Logicomix* [20].*

I noted that the book had been purchased at Robinson Crusoe Books, so on Monday I went there myself and bought a copy. I read it on the bus back to Ankara that afternoon. It was my first graphic novel, and I was impressed; but why shouldn't I be impressed to see a mathematician-philosopher made into a tragic hero like Orestes?

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In class on Tuesday, November 24, Seçil presented VI.8; Tolgay, VI.11; Mehmet, VI.12; and Jeremy, VI.15. Then I

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*`http://www.logicomix.com/en/`

was moved to scold (some of) the students for poor preparation. Seçil had looked at her notes repeatedly. Jeremy was more polished, and he was able to write down the numbers of the propositions that justified the steps of his proof; but he couldn't just explain in words why the steps were justified. I can't fault anybody for having difficulty with the mathematics; but I fault Jeremy for trying to fake his way through a proof. I said to the class that notes were not absolutely forbidden, since we regular teachers did use them ourselves in teaching; still, I said, one ought be able to understand and reproduce the general flow of one of Euclid's arguments without copying from a notebook.

Then Yunus got up and gave an exemplary exposition of VI.16, without notes at all. (He did take a glance at the proposition in my copy of Euclid before proceeding to the blackboard.)

Mürsel followed with VI.17, a special case of VI.16. Then Elif finished the day with VI.18: to construct on a given straight line a rectilineal figure "similar and similarly situated" to a given one. I was sorry she just used a quadrilateral like Euclid, rather than drawing a more outlandish figure to emphasize the generality of the proposition.

* * * * *

For the holiday, Cihan was flying to Bosnia to see his Serbian girlfriend, whom he had met in France. Apparently he didn't get back in time to present VI.19 on Tuesday, December 1; so I presented it.

Rashad did VI.21, which he said was immediate. Nur presented VI.23; but I recall going to the board myself to talk about what "compound ratio" meant. Ali finished our coverage of Book VI with Proposition 24.

You see I haven't too much to report here, in part because

I am late in making the report. But I think the students have been bored; they may think all of these propositions about proportion are obvious. Students have been cutting class too, perhaps to prepare for other classes and their exams. Perhaps they haven't understood that a necessary and nearly sufficient condition for getting a good grade in my course is showing up to class. It's hard to believe, unfortunately, that they don't care about their grade.

Mehmet finished the day with Proposition 1 of Book XI:

> A part of a straight line cannot be in the plane of reference and a part in a plane more elevated.

We discussed whether there was really anything to prove here. Euclid argues that, if the contrary does happen, then the part of the straight line that **does** lie in the "plane of reference" can be extended in that plane. Then two straight lines will have a common segment. I should think this was obviously absurd; but Euclid **proves** the absurdity by drawing a circle with two distinct diameters that have a common endpoint. Mehmet didn't repeat that argument, and indeed the circle doesn't appear in Heath's diagram.

Postulate 2 justifies extending straight lines, but says nothing about planes. It is quite an exaggeration, this modern idea that Euclid builds up his whole system from "axioms." The Modern then has to say that Euclid got it wrong, since propositions like XI.1 are "really" axioms too.

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On Tuesday, December 8, Besmir presented XI.2: In Heath's translation:

> If two straight lines cut one another, they are in one plane, and every triangle is in one plane.

Besmir used Fitspatrick's version, which inserts a parenthesis after "triangle": "formed using segments of both lines." Before starting, Besmir asked if he should write down the proposition, and I told **him** to decide. This was a mistake. He wrote all those words, then he started writing the words of the proof, without saying anything. He said he would explain the proof after he had written it down. I suggested he do these at the same time. He tried; but after some questions from me, he had to admit that he didn't understand the proof. Well, Heath has a note: "It must be admitted that the 'proof' of this proposition is not of any value." There's really nothing that can be proved here, in our sense at least. Euclid's proof does suggest that assumption that two intersecting straight lines must lie in one plane, at least near the point of intersection; then XI.1 can be used to finish the proof.

In any case, it may be of value to confront the students with weird proofs; it may induce them to be more questioning of what they read.

When he was finished, Besmir asked to leave and go study for an exam. I said the door was unlocked.

Elif continued with XI.3: "If two planes cut one another, their common section is a straight line." Despite several attempts, she just couldn't get the diagram right. This reminded me that a three-dimensional imagination may be difficult to acquire.

But Çağdaş gave an accomplished presentation of XI.4.

Taner was supposed to present XI.5, but he was absent, so I did it: If three straight lines are at right angles, at the same point, to another straight line, then the three are in one plane. Both Cihan and Ali raised questions about the argument. I hope they would have raised them also if Taner had been presenting. I think Cihan had not quite understood that (in Heath's diagram) AB, AC, and AF are all in one plane, and angles ABC and ABF are right (which is absurd unless BC and BF coincide). Indeed, the claim cannot be seen from the figure alone.

At the end I set up the proposition explicitly as a converse of XI.4. Together, 4 and 5 are that, if two straight lines meet a third at right angles, and at the same point, and a third straight line meets the others at that point, then this straight is in the plane of the first two if and only if it is at right angles with the third.

With all of this talking, I used up the remaining time.

Today, if we can get through eight more propositions, then we shall be finished with Euclid. Apollonius is next.

1.17. Tuesday, December 15

In class on Friday, December 11, we finished with Euclid. It's too bad, because we stopped with Book XI; but now there are just ten hours left for Apollonius.

Tolgay proved XI.6: straight lines at right angles to the same plane are parallel. Maybe this proposition sets a record for the most auxiliary lines: to prove the parallism of two straight lines, **five** additional straight lines are drawn.

We skipped XI.7, owing to a mistake of mine: I picked the propositions in Book XI by looking at what was needed in Book I of Apollonius, according to the editors of the latter; but I forgot to check which propositions the needed propositions themselves needed.

In fact XI.8 calls on XI.7: a straight line joining points on two parallel straight lines is in the same plane as the parallels. But the latter hardly needs proof—or can hardly be proved, as opposed to being assumed.

Proposition XI.8 is the converse of XI.6, and Tuğba proved it. When she started drawing the diagram, I suggested that she could just use the one left by Tolgay, since it is the same down to the lettering of the points.

[As I noted on p. 44,] Reviel Netz suggests that for the ancients, the diagram is a "metonym" for a proposition; the diagram "individuates" the proposition. By contrast, for us (he says), the **enunciation** of the proposition is the metonym: this is what we quote when we want to specify which proposition we are talking about.

However, XI.6 and 8 have identical diagrams. But in fact, as they are drawn in Heath at least, one is a mirror image of the other: it is reversed. So when I suggested that Tuğba use Tolgay's diagram, she looked at it and decided she had better use her own.

Özge proved XI.9: parallels to the same are parallel to each other. Is it obvious? I don't know about the students, but I think it is not obvious in three dimensions. It's not a **surprising** proposition; but proving it takes a bit of work, and it's "real" work: you take a plane to which the straight line A is at right angles; if B and C are parallel to A, then they are at right angles to the plane by XI.8; then they are parallel to each other by XI.6. And these are the record-breaking propositions in terms of numbers of auxiliary straight lines needed in their proofs. Book XI has some of the same logical music as Book I.

Seçil did XI.10, then we skipped ahead to XI.14, which Yunus did; then Mürsel did XI.15, using and adapting Seçil's diagram from XI.10. I don't know if he was influenced by my earlier comments, or would have done this anyway.

Mehmet did XI.16, and Rashad, XI.18, and that was it.

We had some time left. There were 13 students present, and I wanted to assign the first 14 propositions of Apollonius. Tolga (who has not attended many classes) said he would take the I.1 and 14. Then I just wrote down the rest of the students from left to right. Tolgay said he might miss next class, because of an exam. I said he shouldn't, but I moved him to a later proposition anyway.

I read out Kepler's warning, at the beginning of the Green Lion edition [4] of Apollonius, to the effect that some work is inherently difficult, and Apollonius is an example. Now we'll see what happens!

2. Apollonius and Archimedes

2.1. Tuesday, December 15

Probably it's good that we started today with Proposition I.1 of Apollonius, rather than skip ahead to something meatier. Tolga proved it, as it is proved in the text: I mean, he didn't prove it as if he had thoroughly understood it and was passing on his understanding. Not that there's so much to understand: a straight line joining the vertex of a conic surface and another point in the surface lies on that surface.

The original Taliaferro translation [2] in the Britannica *Great Books of the Western World* introduces a small error, which is repeated in the Green Lion edition: There are three diagrams, showing three possible configurations. In two of the diagrams, B is on straight line AF, not AE, and indeed the text would not make much sense if B were on AE. But in the third diagram, B **is** on AE (extended).

The Heiberg edition does not feature such a mistake.*

Actually, before Tolga started, I wrote down a bunch of Greek terms from Apollonius that gave rise to English words (although the latter may not be the words used to translate

*I printed this out from `http://www.wilbourhall.org/` which has all sorts of old math texts.

the former): The English words were cycle, periphery, parallepiped, epiphany, center, basis, scalene, diameter.

Tolga made the English mistake commonly made [and which I mentioned on p. 74]: He wrote for example "Let C is not on the surface . . . " I asked him to replace "let" with "suppose." But I can't say that the grammatical difference between "let" and "suppose" here is important.

After Tolga, Elif presented I.2. She started sketching the figure, and she said something about "vertically opposite points." She had evidently been confused by the expression, "If on either of the two vertically opposite surfaces two points are taken . . . " I jumped up to try to clarify matters with my own diagram.

Elif worked through Apollonius's proof that the straight line joining the points lies within the surface. I asked if the result was obvious. She said she had thought it was, but on the other hand the proof was a real proof.

Ali proved I.3, that if a cone is cut by a plane through the vertex, the section is a triangle.

Çağdaş asked, Can't the section be just a straight line, as when the cutting plane is tangent to the conic surface?

Ali said, But then the cone would not have been properly **cut** by the plane: the cone is supposed to be cut into two pieces.

Ali asked whether the cone could be infinite, or something like that. I observed that the **cone** has a base, though the **conic surface** can be extended indefinitely, and Yunus would be proving something involving this fact with I.8. (Yunus acknowledged this.)

Somehow I was moved to distinguish the conic surface from the cone by saying the surface was two-dimensional. Ali asked, What does that mean? I think he was teasing me, alluding my own tendency to ask the students what **length** is. We laughed.

Tuğba proved I.4, that a plane parallel to the base cuts the cone in a circle. As she was drawing her figure, I asked whether the proposition was obvious. She said it was. I got up and drew an extremely oblique cone (hers was nearly right) and asked, Is the proposition still obvious? She smiled and said it still was. Nonetheless she did the proof.

One ends up proving $DG = GH = GE$, where H was chosen arbitrarily on the section, so that DGE ends up being the diameter of a circle. Rashad asked whether the "last line" was really necessary; he was referring to the straight line AHK, used to prove that GH is equal to DG and GE. I looked to others for an answer. Ali said in effect that if we didn't have H, then all we can prove is $DG = GE$; but this doesn't establish that G is the center of a circle.

I suggested that, if we just proved $DG = GH$, that would be enough to establish that G is the center of a circle. But again H is a random point, and E is not random: it is in a straight line with DG. Special cases do tend to get special treatment: so the term "ellipse" will not cover the circle, presumably because any curve that is a circle should be called just that.

There were five minutes left, but Seçil said they weren't enough for her to prove I.5 (which may be the first non-obvious proposition). So we stopped.

There were just 13 students present, two who had not come on Friday: Çağdaş and Nur. Çağdaş had the text though; Nur did not. The text is the Green Lion edition, my copy being perhaps the only one in Ankara, except the photocopied pages that the students now have. I asked the library to order a copy, but it isn't in. I do see however that somebody other than myself has asked the library to order: "*Apollonius de*

Perge, Coniques : texte grec et arabe, établi, traduit et commenté sous la direction de Roshdi Rashed; text in Arabic and Greek with French translation of facing pages; introductions, commentaries, and notes in French."

2.2. Saturday, December 19

Last night in Apollonius class, I became sad and depressed about the whole enterprise; but afterwards my belief was restored.

Seçil presented I.5 of the *Conics.* She went about constructing the diagram, but she didn't explain what was special about points G and K. When I enquired, she said there was nothing special about them. So she had missed the whole point, namely that triangle AGK is similar to axial triangle ABC, but lies "subcontariwise." I let her continue with the proof. Eventually she asserted that triangles DFG and KFE were similar, and I pointed out that the missing hypothesis was required for this. I think it was Cihan or Ali who told Seçil where the hypothesis was in the text.

When Seçil wrote down the equation

$$\text{rect.}\, DF, FE = \text{sq.}\, FH,$$

she also wrote the justification supplied by the editors:

> Eucl. III.31, VI.8 porism, and VI.17.

This suggested that she didn't just **see** why the claim was so. I invited her to draw the circle in the plane of the blackboard—the circle whose diameter was DE, to which HF was dropped perpendicularly. She did this, but positioned F as if it were the center of the circle.

When the proposition was finished, so that, in priniciple, we knew that an oblique cone had a circular section in two different directions, I asked Seçil if this was surprising. She said it was.

As I look back at the proposition, I see we didn't remark on the importance of having the plane of the axial triangle be at right angles to the base. In general, students seem to be drawing their cones as if they were right anyway.

Taner proved I.6 very confidently, but he seemed to have relied mainly on the diagrams to tell him what the assertion **was.** He got it wrong. He thought the cone was being cut by a plane through the vertex, making the triangular section AKL; and he thought the base KL was a diameter of the base of the cone.

Ali questioned how the cone was being cut. Perhaps he had understood that the cone was "really" being cut so as to make the axial triangle ABC; or perhaps he was just trying to reconcile Taner's claims with the text, and not succeeding. Taner kept insisting that there was only one cutting plane, making AKL.

Well, AKL **can** be understood as the result of cutting the cone with a plane through the vertex. But that's not how it arises in the text.

I got up and tried to argue this point. Eventually Taner agreed that he had been confused.

While I was up at the board, I saw that many classmates were not paying attention.

Besmir was next with I.7. He got up, drew a diagram, and started writing down words without making a sound. What should I do? The classmates are not Johnnies who will speak up if something isn't going right. They may think it is the **teacher** who should not allow time to be wasted. Eventually

I asked Besmir if he had any teachers who came to class and wrote silently on the board. He just said he needed to write everything down before talking about it.

Well, since (I think) he was working from memory, maybe he needed to concentrate silently. In that case, I would rather he used notes.

By the time Besmir was finished, the usual ten-minute break time was almost over.

I suppose the Apollonius is harder than I think. Unfortunately I can recover no memory of the relevant mathematics tutorials at the St John's. How did I prepare and present propositions? How did others? I can't remember. But I do think that Johnnies were **engaged** in class in a way that most of my students now seem not to be. I think Johnnies understand that they are supposed to be reading every proposition themselves. My students now may study only the propositions that they are supposed to present.

It might be recalled that Johnnies have but one mathematics tutorial at a time. My students are taking other math classes.

By the time they come to Apollonius, Johnnies have spent some weeks or months thinking three-dimensionally with Euclid and Ptolemy. Perhaps my students now have not had so much experience. But Ayşe pointed out later that they **do** have such experience, from vector calculus. From our point of view, some students are just lazy. If I were giving an ordinary lecture class, Ayşe reminded me, most students would not be very engaged in the class, if they bothered to come to class at all. They would cram before exams, and that would be it. Why should I expect things to be different now?

After a late ten-minute break, from which not everybody had returned, Yunus gave a reasonably accomplished presentation of I.8. He answered somebody's question (Ali's, I think) about exactly what was being proved. I asked whether Apollonius was proving that the conic section, which "increases indefinitely," also opens indefinitely wide. Yunus said No, and I guess he's right; the fact will be a consequence of later propositions.

Mürsel was next with I.9. During the break, he had asked me why Apollonius could say "Therefore *GED* is a straight line." I pointed out the ensuing reference to "Eucl. XI.3." Since he didn't have a copy, I handed him my volume III of the Dover edition [21] of Heath's Euclid. Here I recalled the Green Lion remark on the usefulness of a one-volume edition. I had brought volumes II and III to class; but what if volume I had been needed? My Green Lion edition was at home.

Anyway, I don't know why Mürsel was confused, since before the claim in question, Apollonius says "therefore *D*, *E*, *G* are points on the common section of the planes."

At the board, Mürsel took a long time. I gave up hope of getting to I.11 and the definition of the parabola.

Ahmet presented I.10 sheepishly, since it seemed so simple. I asked if he had a modern way to describe the result of the proposition. He didn't, but Cihan offered the word "convex." A conic section is convex: any straight line drawn between two of its points lies entirely inside the section. I went to the board to suggest that the diagram of the proposition was perhaps misleading, since points *G* and *H* on the section were drawn on opposite sides of the axial triangle, but they need not have been. I recalled that Elif had proved the proposition (namely I.2) that justified I.10; she agreed with my memory and observation.

Time was just about up. I already had students signed up for all propositions through I.15; I took more names for the next five. The new names included the students who hadn't shown up to the previous class.

I asked the class what they thought of Apollonius. Was it interesting? Was it just hard? Cihan said it was both. He is one of the better students. He sits at the front and takes notes from his classmates' presentations.

I started talking about how there was no point in doing "math history" unless you read original works. Secondary sources will "modernize" the treatment. If there is really only **one** mathematics, that may be fine; but the unity of mathematics is not obvious.

I don't remember exactly how it all happened, but several of us ended up sitting around for half an hour after class talking about mathematics. The active participants besides myself were Ali, Cihan, and Mehmet, but Elif, Mürsel, and Seçil also stayed around. When the rest of the class was still there, I said something about how Apollonius was rigorous mathematics, whereas there had been periods, as in the 18th century, when math was not about rigor, but was about deriving equations with however-tenuous justification. I had just been reading about Euler's derivation of the value of the sum

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$$

of the reciprocal squares. Because of something he said, I directed a question at Ali: "Do you want to see Euler's derivation?" How could he say no?

I got up to the board, but said class was officially over, so anybody who had to leave should feel free. Those who stayed around seemed to be impressed by the derivation of the value of $\pi^2/6$ for the sum, though they agreed that it was not rigorous. I reported the claim that British mathematics in this period fell behind continental mathematics precisely

because British mathematicians had an excessive devotion to rigor, which kept them doing mathematics in the ancient style. I did mathematics for its rigor, I said; but I had to acknowledge that great advances in mathematics had been made by people who didn't share my interest.

Mehmet made a distinction between mathematics and physics: the mathematician wanted to prove things; the physicist, to discover them. I think those were the words he used. He's majoring in physics as well as mathematics. He talked about physical laws whose mathematicial derivations were not sound. A physicist told him this didn't matter, since the laws agreed with nature. But Mehmet wanted a rigorous derivation of the laws.

I suggested that physics and mathematics had been indistinguishable for much or least some part of history. Ali said that, in the Renaissance, there had been **one** science, and **one** art, and a person like Da Vinci could do both.

We carried on for a while, as I said, until nobody seemed to have more to say. But then Mehmet asked me if there were another edition of Apollonius he could look at, since he had not been able to make sense of his proposition, I.15. I showed him the Greek text of Heiberg with facing Latin translation. I said there was Heath's English version [21], which was not a proper translation; and I explained the origin of the Taliaferro edition. I had the original Britannica version of Taliaferro with me, and Mehmet observed that the diagram for I.15 was completely different there. I don't know why Mehmet would have had a problem with the Green Lion diagram, which is beautiful, albeit anachronistic. I forgot about the English edition by Rosenfeld, which I had once found on line. The English needed editing, and there were no diagrams: the reader was supposed to consult Heiberg for the diagrams. The translator claimed that, as a mathematician, he could correct the deficiencies of Taliaferro.*

Anyway, Mehmet is one of the brightest students I have had; if he is struggling with Apollonius, I suppose that should tell me something. I shouldn't feel disappointed that we are not likely to complete the remaining fifty propositions of Book I of the *Conics* in the remaining seven hours of class!

2.3. Friday, January 8

There have been four hours of class (three sessions) that I couldn't report on. We spent those hours on Propositions 11–15 of Book I of Apollonius.

On Tuesday, December 22, Cihan presented 11, which introduces the parabola. Everything was fine, as I recall. Rashad was then supposed to present 12, introducing the hyperbola, but he was missing, so I had Tolgay go ahead to 13, on the ellipse. He admitted there were things he didn't understand, but I said we would work them out together.

However, there wasn't time to finish in the one hour (rather, 50 minutes) that we had. So Tolgay started over on December 25. He used colored chalk to distinguish parts of the diagram: a good touch. When Tolgay was finished, I asked Rashad if he could draw a new figure, but keep Tolgay's proof on the board, since it was basically the same as the one he needed to give. But no, Rashad had to start afresh.

Apollonius and Euclid may repeat statements in the course of a demonstration. They do not have the modern technique of writing an equation (or rather, proportion), displayed by

*It seems Rosenfeld died last year, but his translation is at `http://www.math.psu.edu/katok_s/Apollonius.html`.

itself on one line, with a number, so that it can be referred to later by that number. But we today can use this technique at the blackboard. Indeed, if a proportion is somewhere on the blackboard, and we want to use it, we can point to it and say "By this we can conclude . . . " I think Rashad was one, but not the only one, of the students whom I tried to convince to use this technique, rather than rewrite something that hasn't yet been erased. He still wanted to rewrite. Such students are at the stage of following an argument of Apollonius step by step, without seeing it as a whole.

Tolga presented 14, on the opposite sections, in the manner I have come to expect: he sounds fairly polished, but he may not really know what is going on. Afterwards, I tried to emphasize the point of the proposition: No matter how oblique your cone or rather your "opposite surfaces" are, no matter whether your cutting plane cuts one surface near the vertex, and the other far away, you get the same section from either surface.

On December 28, Mehmet presented 15, the finding of a second diameter of the ellipse. After his successful conclusion, I admitted that the proposition was still mysterious to me, although it became unsurprising if one wrote things out with "Cartesian" coordinates. I had written out my own rearranged and streamlined argument, but I didn't take the time to show the students. (The point is that most of the argument can be written out as a chain of equal ratios, as in $A : B :: C : D :: E : F :: G : H :: \dots$)

I said that an ellipse by definition is a certain kind of conic section; by demonstration, the ordinates of an ellipse have a certain relation to the abscissas. Proposition 15 shows that an ellipse has a second diameter, with respect to which new ordinates have a similar relation to new abscissas. But this does not show that there is a cone that would give us that second diameter along with the ellipse: showing this would take a lot more time and propositions.

There is a remarkable point in the demonstration where Apollonius takes the difference between one area, say A, and another area, say B, although B is not actually a part of A. We know that B is **equal** to a part of A; but still, to speak of the difference between two disjoint areas suggests the idea of an area as something **abstracted** from a figure.

But I hadn't yet checked the Greek text. Apollonius doesn't speak of "A minus B"; he says "A exceeds B by C."

Time was up. Unlike December 25, January 1 was a holiday, so our next scheduled class was to be January 5. However, I stayed home with a cold that day. It wasn't a matter of life or death, but I just didn't feel like putting in the effort of making my way to the university. In fact, perhaps staying home didn't improve my rate of recovery; I still feel worn out by the cold, though I am in the office getting ready for the last class of the semester.

I did spend time over the weekend thinking of what might be done next semester. There are passages of Al-Khwarizmi, Al-Uqlidisi, Thabit ibn Qurra, and Omar Khayyam that are worth reading, along with Cardano, Descartes, Newton, and Lobachevski. Unfortunately, if there is not enough time, it is the first four, not the last, that should be jettisoned.

2.4. Friday, January 8

The last class of the semester is over.

On Tuesday, December 28, I got an email from Melis, who was scheduled to prove Proposition 16. She was however writing from her home in İzmir (Smyrna), whither she had made a snap decision to go. It is common for students to make the holidays longer than they are officially scheduled to be.

We did Proposition 15 on Tuesday, but there was no time for 16 anyway. If there had been, I may have proved it myself, more efficiently (in my view) than Apollonius does. The proposition is to me a rare example where a proof by contradiction is better than a direct proof.

So today we opened with Melis's proof of 16: that an hyperbola has a conjugate diameter. But Melis didn't actually give **her** proof. She went up with her copy of the Green Lion text and started copying its contents onto the board. After a while I asked her what she was trying to prove, since she hadn't made it clear.

"I don't know" she said.

Another dilemma for Teacher. I could be a disciplinarian and send Melis to her seat, with a reminder that nobody should write down anything that *she herself* doesn't understand and believe. But then I should have been doing this throughout the semester. In the event, I told Melis what she was supposed to be proving.

However, when Melis continued copying things down from the text, I questioned this practice, noting that we all had the text and could read it for ourselves; she was supposed to be **explaining** to us, I said. She made some weak attempt at this, but it was more like explaining to herself.

Near the end, Apollonius makes a leap that the editors justify with a footnote. Melis ignored the footnote, and Cihan asked why the leap was justified. I don't **think** he was testing Melis; he really wanted to know. Ali attempted an explanation, and then it appeared that **he** had missed the point of the proposition: he thought $AK = BL$. This is true, but it is precisely what must be proved.

Melis finished somehow, and then I got up to offer my proof by contradiction of the same result, as well as a general comment on the import of the proposition.

Nur did 17, a simple but confusing proposition; and Nur indeed was confused. I went to the board and discussed the situation with her. Actually I'm not sure how we know that point C exists (under the assumption that leads to contradiction). I said this.

Ahmet did 18, and Çağdaş, 19. Taner was supposed to do 20, the next proposition after 16 that involves actual lengths. But he was absent, so I proved this, along with 21. (The last had been unassigned, but I thought we should finish with this rather than 20.)

Then, in a semester course on ancient mathematics, I spent half an hour talking about Archimedes. I gave his rigorous quadrature of the parabola, then mentioned the non-rigorous version on the Archimedes Palimpsest discovered in Istanbul by Heiberg a century ago.

Then I had to stop. But before class I had written on the board the names of the mathematicians I wanted to read next semester. Ali at least was interested and wanted to know how to get a hold of the texts for a friend.

In the break, Özge asked about the final exam, and in particular whether they still had to know the Greek alphabet. I said Yes. She complained that she didn't want to memorize it again. I said that most of her classmates had not bothered to do this on the first exam, so I thought it was fair to ask again; but I said I would re-memorize the Russian alphabet (which I had learned for one of the two language exams for my doctorate, though I forgot everything soon after the exam).

Part II.

Spring semester

About the course

This is from the Math 304 webpage:

This course is a continuation of Math 303, but that course is not a prerequisite for this one. Practices will be as in Math 303:

- attendence is required;
- all students will spend time making presentations at the blackboard;
- there is no "textbook."

This course will make no attempt to fit the catalogue description. Some phrases in that description are apparently based on chapter titles in Boyer's *History of Mathematics.* But again, this course will not follow a textbook; we shall read original sources (albeit in translation, from Arabic, Latin, French, . . .). This approach is slower, but more honest to the title of the course. Why?

- I accept the conclusion of the philosopher R. G. Collingwood [see Appendix C] that history is the history of thought. This means, in particular, that doing history of mathematics means thinking the mathematical thoughts of past mathematicians.
  - This is difficult work, but nobody else can do it for us.
  - This work can hardly be done without looking directly at what these mathematicians actually wrote.
- Second-hand accounts of past mathematics may give a misleading view, as for example by translating everything into modern algebraic terms.

Anybody who is interested can read a conventional "history of mathematics" on their own. But there is no substitute for working together, as a group, to understand some old piece of original mathematics.

Some students took Math 303 in hope of learning some history in the sense of stories. The words "history" and "story" are indeed cognate, coming through French from the Latin *historia,* which is from the Greek *ἱστορία*. However, we know almost nothing about the personal lives of ancient mathematicians. About more recent mathematicians, more is known. For example, there is this interesting piece of information:

> After his death, Newton's body was discovered to have had massive amounts of mercury in it, probably resulting from his alchemical pursuits. Mercury poisoning could explain Newton's eccentricity in late life.*

This is irrelevant to the understanding of Newton's mathematics (though it might be used as an excuse for not understanding Newton).

Some students in Math 303 were disappointed in the quality of some of their classmates' presentations. However, student presentations are essential to this course. You don't really understand something unless you can stand up and talk about it. Also, in this course, everybody should have read what is being presented at the blackboard, and everybody should be prepared to criticize a faulty presentation, or to raise questions.

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*`http://en.wikipedia.org/wiki/Isaac_Newton`, accessed February 17, 2010

3. Al-Khwārizmī, Thābit ibn Qurra, ‘Umar al-Khayyāmī

3.1. Thursday, February 18

There were 20 students in class; three of them were not among the 37 registered students. I discussed what is on the webpage.* I stated Propositions 5 and 6 of Book II of Euclid's *Elements* [23] and I drew the diagrams that prove the propositions. In algebraic notation, the propositions are:

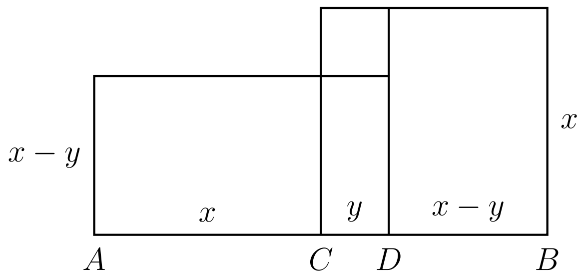
$$\begin{aligned} (x+y)(x-y)+y^2 &= x^2 \qquad \text{if } y<x, \\ (x+y)(y-x)+x^2 &= y^2 \qquad \text{if } x<y. \end{aligned}$$

See Figure 3.1. Written algebraically, the propositions become the "same" if we switch x and y in the second line.

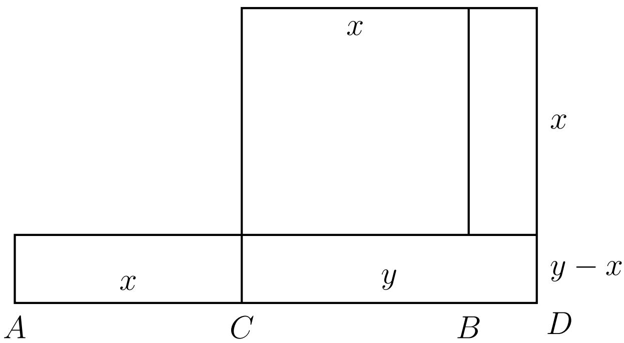
But Euclid doesn't write things this way. I introduced the propositions by asking:

> If a straight line is to be divided in two, where should the point of division be chosen so as to maximize the area of the rectangle bounded by the two pieces?

*http://metu.edu.tr/~dpierce/Courses/304/



(a) $(x + y)(x - y) + y^2 = x^2$



(b) $(x + y)(y - x) + x^2 = y^2$

Figure 3.1. Quadratic equations as in Euclid

Ali answered (he was one of the best students in last semester's course). He said that the straight line should be bisected.

"Why?" I asked. He observed that if the point of division approached one of the ends of the line, then the rectangle would become small.

I thought this a reasonable way to think of the problem. But then I have a memory of thinking this way as a child: I was playing with a rubber band, and I wondered if the area enclosed by the band remained constant through all possible contortions of the band (in a plane). The answer was obviously "No," if one observed that the band could be straightened out so as to enclose nothing.

My maximization question in class on Thursday was one that may come up in a calculus class. But I don't think anybody should be impressed at the ability of calculus to answer the question, since the answer is so easily found without calculus. Indeed, if you divide the line equally and unequally, then Euclid's II.5 shows by *how much* the rectangle bounded by the equal parts exceeds the rectangle bounded by the unequal parts: it exceeds by the square on the line between the two points of section.

Euclid's II.6 is about what happens when the line is divided "externally." Euclid doesn't use this language, and I don't know whether he thought of it. Me, I am delighted to find that two propositions are just instances of one idea; but I can only guess whether Euclid sought such delight. (Presumably he saw that II.5 and II.6 were intimately related; but I don't know what he thought the relation was.)

Again, Proposition II.5 is that, if straight line AB is bisected at C, and D is chosen elsewhere on AB, then

$$\text{rect.}\,AD,DB + \text{sq.}\,CD = \text{sq.}\,AC.$$

Proposition II.6 is about what happens when D is chosen on the *extension* of AB beyond B. Then

$$\text{rect.}\,AD,BD + \text{sq.}\,AC = \text{sq.}\,CD.$$

These become the same proposition if we use "directed" lines and allow "negative" areas, so that rect. AD,DB is the "negative" of rect. AD,BD. But I have no reason to think that Euclid considered this possibility.

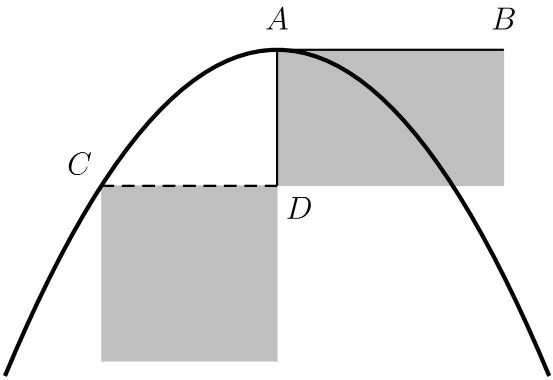


Figure 3.2. Parabola

In the remainder of class, I started to state what we would need to know about conic sections in order to understand Omar Khayyám's solution of cubic equations by means of conics.

Specifically, I said that a parabola has an *axis* and a *parameter.* Suppose the parameter is AB, and the axis is AD, drawn at right angles. If C is chosen on the parabola itself, and CD is drawn at right angles to the axis, then

$$\text{sq.}\, CD = \text{rect.}\, AB, AD,$$

as in Figure 3.2. An ellipse (Figure 3.3a) or an hyperbola (Figure 3.3b) has an axis AB and a parameter BC so that, if D is chosen on the curve, and DE is dropped at right angles to the axis (or the axis extended, in the case of the hyperbola), then

$$\text{sq.}\, DE : \text{rect.}\, AE, EB :: BC : AB.$$

I postponed till another time the definition of proportion. After class a student (possibly Mehmet Arif Şekercioğlu) asked

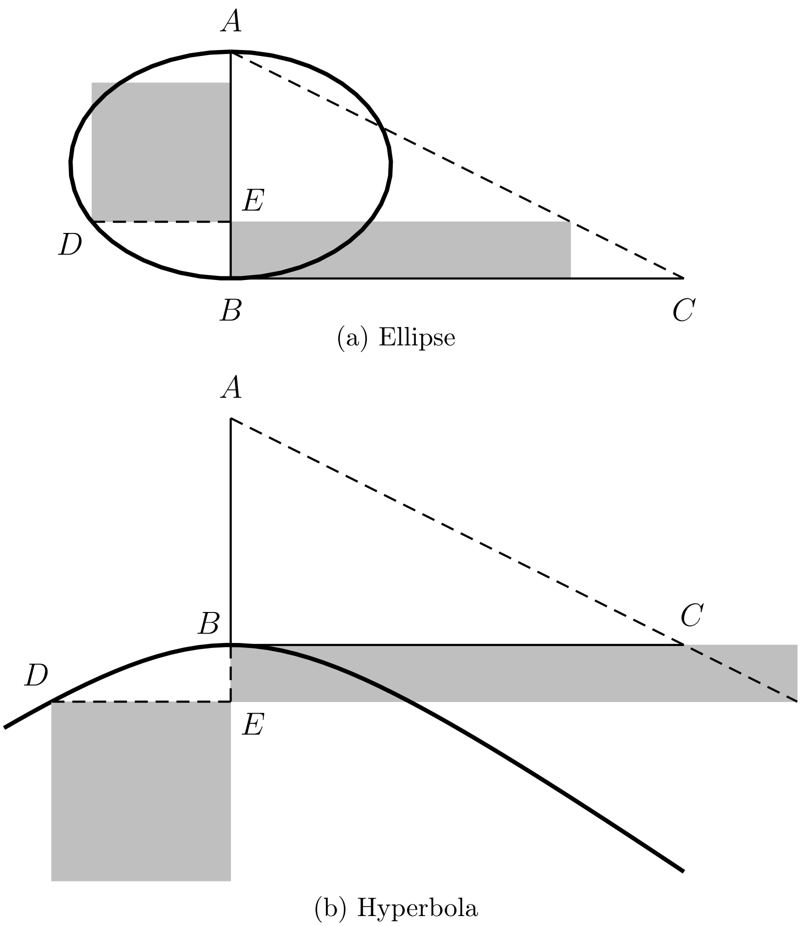


Figure 3.3. Ellipse and hyperbola

for clarification of the definitions of conic sections. I sketched a cone as in Apollonius [4] and said that he proved that, if you cut the cone, the sections had the properties I described. Maybe I'll say this to the whole class later.

I did intend to be a bit intimidating in the first class, trying to ensure that only committed students stayed with me beyond Add-Drop Week.

3.2. Tuesday, February 23

Perhaps I shall not have a problem. Twenty-three students showed up—more than last time, but not many more. (Seven from Thursday did not return.)

I had told the students on Thursday to read the selections from al-Khwārizmī and Thābit ibn Qurra (taken from Katz [35]) that I had put on the webpage. I said the students should be prepared either to explain these passages or say why they didn't make sense. But when I came to class on Tuesday, it appeared that only one student (Zhala) had actually printed out the selections. Another student (Oğuzhan) had read the selections on the computer screen and taken notes; he said he could expound their contents. Perhaps others had done something similar; but to find out, I should have had to interrogate them one by one.

Instead of doing that, I gave the book to the nearest student (Dilber) and asked her to read the first paragraph of al-Khwārizmī; we discussed this, then another student read the next paragraph, and so on.

The first paragraph (after the preface invoking the blessings of the deity) seems to allude to "Arabic" numerals. That's what one student said, and I agreed, saying that if we had more time it could be fun to read Al-Khwārizmī's exposition of the Hindu base-ten numeration system: I gather this exposition is the reason why we call them Arabic numbers. I wrote on the board

1 2 3 4 5 6 7 8 9

and asked what one calls these in Turkish; the students said *rakam.* Then I wrote

I II III IV V VI . . .

They told me these were *Romen rakamları.* There seemed to be some awareness that English uses the term "Arabic numerals" for the former; but in Turkish they are just numerals.

Al-Khwārizmī introduces squares, roots, and numbers. But they are all numbers. His first example is

Square is equal to five roots of the same.

With student approval, I wrote this as

$$x^2 = 5x.$$

(Let me just say once for all that when I write such things, I periodically recall that our authors do *not* use such language.) Al-Khwārizmī then concludes

$$x = 5.$$

I asked if there was any problem here. Somebody said $x = 0$ was another solution; but it seemed to be agreed that this was of no interest.

When al-Khwārizmī got to the more complicated example—

one square, and ten roots of the same,
amount to thirty-nine dirhams [$x^2 + 10x = 39$]

—I had Oğuzhan go to the board and present al-Khwārizmī's cookbook solution. It is a solution that in my opinion is not self-justifying: it arrives at the answer 3, and one can check that this is correct, since three squared plus ten times three is indeed thirty-nine; but one does not know *why* this should be correct.

More on this later. The students seemed to understand that al-Khwārizmī's "dirham" just meant a unit. Ali knew that it had been in particular a unit of weight. I observed that it was still the monetary unit of Morocco and that it derived from the Greek δραχμή.

Meanwhile I had Zhala go to the board to write out the solution to

square and twenty-one in numbers
are equal to ten roots of the same square [$x^2 + 21 = 10x$];

Yasemin read out the steps of the solution as necessary. Here two solutions arise. Why?

Well, al-Khwārizmī does go on to give a geometrical justification. For this, I had Murat go to the board to draw the diagrams, while somebody else—Salih Kanlıdağ, I think—read out the steps.

Murat's full name is Murat Yaşar Kurt, but he told me likes to be called MuYaKu ("like Japanese" he said). He turned out to have a printout of the text as well. He was not particularly prepared to draw al-Khwārizmī's diagram; but he worked it out.

So now we had two solutions of $x^2 + 21 = 10x$ on the board: the "arithmetic" solution that Zhala had written, and the geometric one that MuYaKu had written. Some students agreed with me that the geometric solution was at the same time a

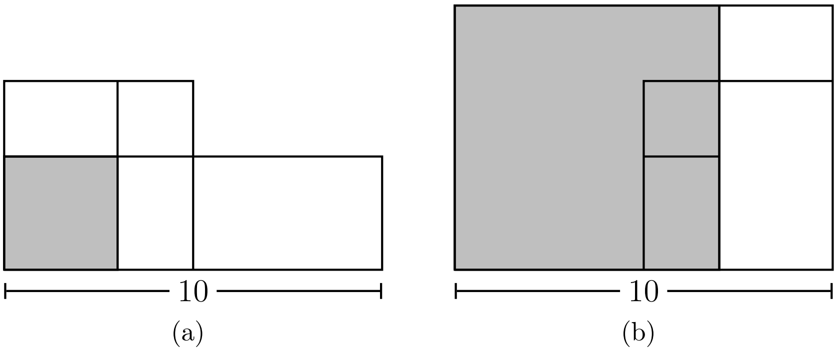


Figure 3.4. A quadratic equation as in al-Khwārizmī

proof that it was a solution. But MuYaKu said they were both proofs, just done in different styles.

I wrote out the geometric argument more quickly, arriving at the answer 3. What you do is draw a square, then extend one of the sides to have total length 10 units; see Figure 3.4a. You complete a rectangle next to the square and on the extension of the side. The rectangle is supposed to have area 21 units; and this with the square makes 10 "roots." Now bisect the line of length 10. This has already been divided unequally, and the rectangle formed from the two pieces has area 21, as we said. By Euclid II.5, the square on 5 exceeds 21 by the square on the line between the two points of division. So this line is 2 units long, and the original square has side 3 units long. I didn't actually refer to Euclid; we in effect reproved the proposition. Anyway, 7 is also a solution to the original problem: why didn't this come directly out of the geometric argument?

Oğuzhan knew the answer: al-Khwārizmī's drawing assumes that the midpoint of the line of length 10 lies beyond the side of the original square. If it lies inside, we get 7. See Figure 3.4b.

Time was about up. Al-Khwārizmī considers three kinds of problems:

1) square and roots equal a number,
2) square and number equal roots,
3) square equals roots and number.

As an **exercise,** I suggested working out geometric solutions to the remaining cases, as for example in the following instances:

$$x^2 + 10x = 39, \qquad x^2 = 4x + 21.$$

Probably Al-Kharizmi does this himself in the full text (which I linked to on the webpage; I didn't want to use those versions in class though, because they are full of footnotes explaining things in symbolic terms).

Oğuzhan had indicated that al-Khwārizmī was solving equations

$$ax^2 + bx = c.$$

I agreed, but observed that he didn't use unspecified coefficients like a and b.

I think Thābit ibn Qurra does in effect use general (unspecified) coefficients; but this will be our topic for Thursday's class.

3.3. Thursday, February 25

Indeed, Thābit ibn Qurra gives a geometric solution to the problem:

māl and roots equal a number.

A note says *māl* is the Arabic for *asset.* I am embarrassed to recall only now that Turkish has borrowed the word with

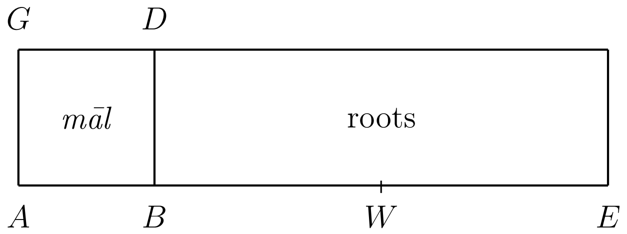


Figure 3.5. Thābit's proof

this meaning. No student pointed this out; is that because the point is obvious to them, or because they didn't notice it?

In any case, the meaning here is "square," that is, "square of the root." Thābit ibn Qurra draws a square $ABDG$ (but he calls it $ABGC$) and extends it by a rectangle that represents the "roots"; the whole rectangle then is the "number."

Say one of the long sides of the large rectangle is AE; this contains B (Figure 3.5). Let AE be bisected at W. (So Thābit introduces letters in the order $ABGDEW$, at least in translation; was he using Greek letters, including the digamma? Ali suggested it might be so, though I don't know if he knew about the digamma. Would Arabic letters be transliterated thus too? No student claimed knowledge of these letters.)

By II.6 of Euclid's *Elements,* to which Thābit refers explicitly,

$$\text{rect.}\, EA, AB + \text{sq.}\, BW = \text{sq.}\, AW.$$

But rect. EA, AB is the given "number," and BW is half the given number of roots, so sq. BW is known; hence sq. AW is known; hence AW is known. The claims about what is "known" allude to Euclid's *Data* [24], though only the editor's footnote makes this explicit. Finally, AW minus BW is known; but this is the desired root.

So the original equation is soluble in principle. And this claim holds generally. Thābit ibn Qurra's alternative to using literal constants in an equation like

$$x^2 + bx = c$$

is to make the equation into a picture. We just somehow understand that one picture can stand for many cases; to suggest otherwise is to suggest that, even if we know how to solve $x^2 + bx = c$, we are not sure we can solve $x^2 + dx = e$.

It is worth noting that Thābit ibn Qurra does not actually give a construction for solving the equation; he just shows that it *can be done.*

Again with the passage of time, I've forgotten who presented the above solution in class. In the excerpt in the book we've been using, Thābit ibn Qurra goes on to solve the equation

māl and number are equal to roots.

I decided to skip doing this in class, in order to review conic sections again, as they would be needed for Omar Khayyám's solution of cubic equations next time.

3.4. Tuesday, March 2

I asked if somebody could present Omar Khayyám's solution (also in Katz [35]) of an equation of the form

cube and number are equal to sides.

(So "side" is what we called "root" before.) Several students said they hadn't been able to follow the argument. Mehmet volunteered to go to the board; but first I got Gökçen to read the selections from Khayyám's introduction that are included in the text. Some key points:

1. Khayyám says you gotta know Euclid's *Elements* and *Data,* along with the first two books of Apollonius's *Conics;* but that's enough.
2. There are four geometric "degrees": (absolute) numbers, sides, squares, and cubes; you can talk about square-squares, but only "metaphorically."
3. Only equations involving numbers, sides and square can be solved *numerically,* so far: perhaps somebody in future can do more. Khayyám's solution of cubic equations will be *geometric.*
4. The numeric/geometric distinction was recognized by Euclid; why else would he develop a theory of ratios of *magnitudes* in Book V, then an independent theory of ratios of *numbers* in Book VII?

Mehmet then worked out Khayyám's solution of the equation above. It involves a parabola and an hyperbola: their point of intersection determines the solution. Mehmet rewrote the equation symbolically as

$$x^3 + a = bx.$$

During the course of things, I asked: Why must the parabola and hyperbola intersect? Somebody, I think Fuad, said they need not.

Indeed, Khayyám notes that the curves might be tangent, or meet in two points. But he doesn't give conditions for tangency. I suggested this as an **exercise** for the students.

It is too bad most of the students were not with me last semester to read Apollonius. I just told them that Apollonius shows how conics can be found with given axes and parameters, and this justifies what Khayyám does. But it's not a ruler-and-compass construction; indeed, one needs a third dimension for the cones themselves.

I observed that if $x^2 + a = bx$, then x is half of b plus (or minus) the square root of the sum of a and the square of half of b. I did this geometrically, but got confused, so the students helped me out. I asked how we could *construct* a square root, and Fuad came to the board to do this with a circle, though he was a bit hesitant. In any case, there are algorithms for extracting square roots numerically. (The anthology of texts has an algorithm for fifth roots, but I skipped it.)

I observed that we didn't have a way to convert Khayyám's solution of the cubic into a similar construction and method of computation.

After the break, I proposed a way to symbolize Khayyám's work that was closer to what he did. Khayyám lets AB and BC be at right angles as in Figure 3.6, and the square on AB is equal to b and is the base of a solid whose height is BC, the solid itself being equal to a. Thus, in Cartesian style, if we let

$$AB = c, \qquad BC = d,$$

then we have

$$a = c^2 d, \qquad b = c^2.$$

Treating BC as x-axis and AB as y-axis, we draw the parabola DBE and the hyperbola ECZ given respectively by

$$cy = x^2, \qquad y^2 = x \cdot (x - d).$$

The parameter of the parabola is c; of the hyperbola, d, and this is equal to its axis. The equations give us

$$\frac{c}{x} = \frac{x}{y} = \frac{y}{x-d},$$
$$\frac{c^2}{x^2} = \frac{x}{x-d},$$

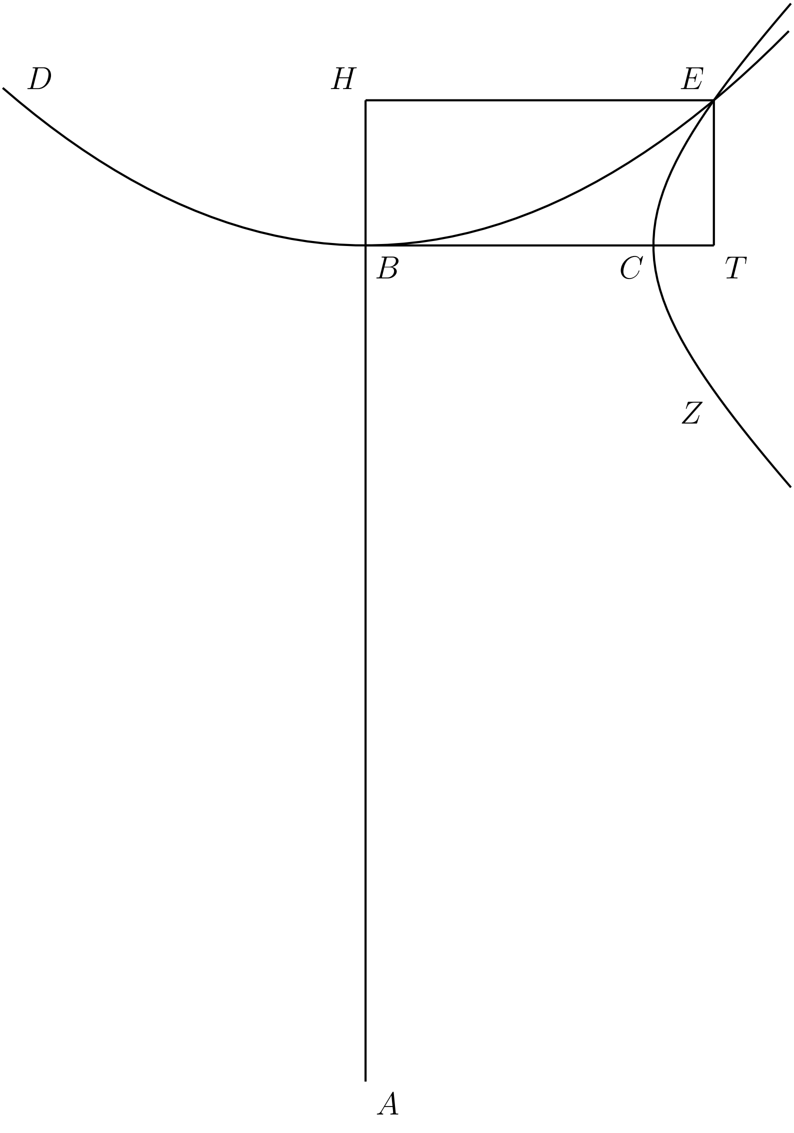


Figure 3.6. Khayyām's solution of a cubic

$$x^3 = c^2 \cdot (x - d),$$
$$x^3 + c^2 d = c^2 x,$$

as desired. This derivation follows Khayyám's verbal description pretty closely, I think. But I have no intuition for actually coming up with this solution. Well, that's what I said in class anyway; but now that I think of it, I see that the steps of the algebraic derivation are pretty easily reversible.

But did Khayyám think this way? I don't know. I also don't know whether Khayyám's solution is original with him; I think Greek mathematicians knew how to find cube roots with conic sections, anyway.* But again, Khayyám refers explicitly to Euclid and Apollonius; if he were using additional old work, he might have said so.

In the reading, Khayyám also solves

cube and sides equal squares and number.

I worked through the solution myself (it uses an hyperbola drawn with respect to given asymptotes, and a circle). Then I took volunteers for presenting the several sections of our next reading: Chapters I, II, VI, XI, XXXVII, and XXXIX of Gerolama Cardano's *Ars Magna* or *De Regulis Algebraicis.*

The book opens with an attribution of the invention of the art of algebra to Muhammad the son of Moses the Arab,—that is, Al-Khwārizmī. It gives a "numerical" solution to cubic equations. Anthologies include this, but it's not much fun to read out of context.

At least Struik's anthology [50] has a fairly literal translation. (I don't remember what Smith's [46] is like.) The whole

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*They did [I add in 2018], according to the notes of Eutocius on the cube-duplication problem that accompany his notes on Archimedes [6].

of Cardano's book was translated by Witmer in 1968 [12], but Witmer freely uses modern notation. This helps one read, but is misleading. The original Latin can be found on the web: I found it through *Wikipedia.* Unfortunately this is the text from a posthumous edition of Cardano's complete works from 1663. Unfortunately, as Witmer says, each edition of the *Ars Magna* kept the old mistakes and introduced new ones.

Over the weekend I started typing out some sections of the Latin, with parallel translation: Witmer's or Struik's translation, with some adjustments by me. But this job became too daunting as the amount I wanted to include grew. So I just gave students copies of sections of Witmer's translation, along with the Latin from the `pdf` file on the web, in case they want to look at that (and Mehmet D., at least, said he did).

4. Cardano

4.1. Thursday, March 4

Ali and Emir presented sections 2–5 of Chapter 1 of Cardano's *Ars Magna.* The two didn't know each other before signing up for this assignment on Tuesday. I think that, rather than working together, they just decided to alternate sections. Ali began, talking about sections 1 and 2; then Emir continued with 3, and so on.

When Ali started speaking, there was still a bit of chitchat in the room. I asked him if people were paying attention to him. He didn't seem too concerned; but then I shut people up.

During Ali's and Emir's presentations, I occasionally went to the board to make a point. I could see that students were asleep or had blank expressions. At the end of class I said they all looked like Zombies, and they laughed.

But then MuYaKu came to me and asked, "What exactly did we do today?" I said, "You should have asked this earlier!"

I thought the reading was fascinating, though on the surface it appears trivial. Superficially, the reading is a discussion of negative numbers: for example, 9 has two square roots, namely 3 and -3.

But this is a misleading account, which is unfortunately encouraged by Witmer's translation. There is no "-3" for Cardano; there is 3, considered as minus. Something like that. There is no symbol -3; there is just "m̃.3." or "3. m̃." (Possibly the periods are just thrown in by the typesetters. The tildes are presumably to indicate an abbreviation.)

Later in the book, Cardano will suggest the possibility of taking the square root of a negative number. Here he ignores this. Hence for example 81 has only two fourth roots, namely 3 and minus-3; but there could be two more, namely the square roots of minus-9.

To be continued in §4.3.

4.2. Excursus on negatives and cubics

By a Johnnie reading the foregoing, I was asked:

> Do you have any suggestions [for a seminal text on negative numbers]?

I answer: Texts other than what we read at St John's are new for me as well. In his *Mathematical Thought from Ancient to Modern Times* [37], Kline says,

> One of the first algebraists to accept negative numbers was Thomas Harriot (1560–1621), who occasionally placed a negative number by itself on one side of an equation. But he did not accept negative roots.

What does this mean? Did Harriot actually write equations in the modern symbolic sense? This seems to be one more example of why a math history book is not of much value in isolation from the original texts.

In any case, Cardano was dead before Harriot was born, but Cardano had given some recognition to negative numbers, as I have said.

Kline makes another strange comment earlier in his book (page 192):

> In arithmetic the Arabs took one step backward. Though they were familiar with negative numbers and the rules for operating with them, through the work of the Hindus, they rejected negative numbers.

By the way, leafing through Kline's book, I notice something that is apparently wrong. (I find it worthwhile to collect such examples, in case one of my colleagues in future still wants to teach the history course out of a modern textbook.) Kline says (p. 194),

> As for the general cubic, Omar Khayyam believed this could be solved only geometrically, by using conic sections.

But Khayyam says (in the translation in the Katz book),

> But, as for the proof concerning these kinds [of equations], if the subject of the question is simply a number, neither we nor any of the algebraists have been able to do it except in the first three degrees: number, thing, and *māl*. But perhaps someone else, who will come after us, will know [how to do] it.

Maybe Khayyam contradicts himself somewhere else on what the future may hold, but I don't know why he would.

My correspondent replied:

> Perhaps Klein is right. I believe the case is that the general cubic cannot be solved by an algorithm when there are 3 real roots, the so-called "irreducible cubic." I have not checked

this lately, but if it is so, then Khayyám may be intending to say that some, not all cubics can be solved numerically. It is these irreducible cubics, by the way, that Viète solves geometrically in props 16–18 of the *Supplementum Geometriae.*

I answered this as in §4.4.

4.3. Thursday, March 4, again

For Cardano (still) there are three kinds of quadratic equations:

1. square and roots equal to a number,
2. square and number equal to roots,
3. square equal to roots and number.

("Root" here is *res* "thing," though Witmer doesn't like the translation "thing"; he just uses x, though Struik uses "unknown.")

Type (1) has one solution (*æstimatio,* I believe)—that is, one *positive* solution. For us there is also a negative solution, but not, apparently, for Cardano: for him the equation

$$x^2 + 3x = 28 \tag{$*$}$$

has just the solution 4. But the funny thing is, this means for Cardano that the equation

$$x^4 + 3x^2 = 28 \tag{$\dagger$}$$

has two solutions, 2 and minus-2: but these are also "equal to each other" for Cardano, rather as in "equal and opposite," I suppose—and this is a reason not to write "minus-2" as "−2," since -2 is obviously different from 2.

Cardano doesn't write out equation ($*$), only (†); so this is all Emir wrote in his presentation. I asked Emir how he knew that (†) had the solutions 2 and minus-2; in reply, he wrote out ($*$) (with t instead of x, since he let $t = x^2$); then he solved it. But he did this by transforming it into

$$t^2 + 3t - 28 = 0$$

and then observing $28 = (4)(7)$ and $3 = 7 - 4$, so that the roots are 4 and -7.

I said, "So you're factorizing," and Emir agreed. In other words, Emir was finding what I would have written as

$$t^2 + 3t - 28 = (t + 7)(t - 4).$$

But he didn't actually write out the factorization this way; he just wrote down the 4 and the 7. To write more would have been against his training to find answers as quickly as possible and fill in the right circle on the multiple-choice answer form supplied with the national university entrance exam.

Apparently Emir has not picked up on the geometric solutions we have worked out, whereby one finds that t is the square root of the sum of 28 and the square of half of 3, minus half of 3. This raises for me the question of whether to encourage the students more to try to think in the old-fashioned way.

Cardano works out similar examples with types (2) and (3): Type (2) has either two or no solution, so the corresponding quartic (with x replaced with x^2) has either four or no roots. Type (3) has one [positive] solution, so the quartic has two solutions.

How do we know that these solutions exist? Ali observed at some point that Cardano seemed to be making some sort of continuity assumption. I said that we had a geometric construction of solutions of quadratic equations. But Ali seemed to understand "geometric" as "physical": we could obtain a line segment as the solution, but our measurement of this segment would yield a rational number, even though the correct solution might be irrational. Ali mentioned that the Pythagoreans knew about the irrational, and that this caused a crisis for them; I passed on what I had learned from Mr Thomas on the J-list, that there was no evidence of such a crisis.

I also observed that Cardano was going to be using cube roots, even though there is no ruler-and-compass construction of these. But I asked whether anybody knew an algorithm for extracting square roots. Nobody did. My father had once told me that he had learned such an algorithm, and a couple of years ago I derived an algorithm for myself for some reason, while teaching a number-theory course. One of the Arabic readings that I skipped in the Katz book concerns extraction of a fifth root. So I suppose Cardano believed in roots because they could be calculated (albeit only approximately).

But Cardano observes further (and Ali presented this part) that if any number ("even a thousand") of odd powers are "compared with" (that is, equated to) a number, then there will be one "true" solution, but no "fictitious" [negative] solution. This is the most remarkable statement in the reading. Ali understood its import, but I don't know if everybody else did. (As I said, they were zombies at this time of day, this late in the week. Maybe I should make tea for them, as Ayse and I did one year when each of us was teaching a Saturday class, to mostly the same students.) If we have the equation

$$ax + bx^3 + cx^5 + dx^7 + \cdots = N, \qquad (\ddagger)$$

then the left side increases from 0 as x increases from 0; also the left side grows without bound as x does; "therefore," for just the right value of x, the left side will be exactly N.

Perhaps it's not hard to accept this. There's a puzzle that goes something like, If you drive your car at varying speeds 300 miles in 5 hours, must there be a 60-mile stretch that you cover in exactly one hour? The answer is supposed to be Yes, because if you let $f(x)$ be the time required to travel between the x-mile and $(x+60)$-mile points, then $f(x)$ will sometimes be above, sometimes below one hour, so for "some" x it will be exactly one hour. But this makes an unjustified continuity assumption.

In *Mathematical Thought from Ancient to Modern Times,* Kline writes [37, p. 198]:

> Perhaps most interesting is the Hindus' and Arabs' self- contradictory concept of mathematics. Both worked freely in arithmetic and algebra and yet did not concern themselves at all with the notion of proof.

We may just as well refer to Cardano's "self-contradictory concept of mathematics." What is the real proof that (‡) has a solution? Cardano shows no sign that this is a reasonable question.

Cardano discusses also the signs of the roots of cubic equations. He states without proof that the equation

$$x^3 + a = bx$$

has no, two, or three roots, depending on whether two-thirds of b times the square root of one third of b is less than a, equal to a, or greater than a. Emir just reported the rule, giving no indication of having thought about where it came from.

Cardano gives no indication of its origins either. (I should say that Emir was reporting the symbolic formulation of things, as recorded in Witmer's footnotes, and not Cardano's verbal formulation. This is another reason not to like Witmer's edition.)

If there are two roots, then one is negative and is "twice" the other (that is, it is minus-2 times the other). If there are three roots, then one is negative and the sum of the other two. Witmer has a long footnote about whether Cardano understood the meaning of this: in the two-root case, there are "really" three roots, but the two positive roots are identical.

I pointed out to the class that we could understand the situation from looking back at Khayyam's geometric solution. He solved this case of cubic with a parabola and an hyperbola with axes at right angles to each other. If these curves do not intersect, there is no solution; if they are tangent, there is one (positive) solution; in the last case, the curves intersect twice, giving two positive solutions.

Time was up.

As I said, MuYaKu came to me after class, asking what exactly we had accomplished. I don't remember exactly what I said, but I talked about Khayyam's solution, which was still there on the board. MuYaKu said this wasn't a solution. I think he meant that we didn't really "have" the line that, according to Khayyam, is the solution. I asked him whether we "had" the square root of 2.

Before MuYaKu talked to me, but after class was over, Ece asked about what student presentations were supposed to be like. She evidently *had* been reading Cardano and had observed all of these unjustified statements; was she supposed to find justifications for them?

Good for her! I said the readings were not the word of God. Not everything needed presentation; the presenter should decide what was important. If there were unjustified claims, the presenter should say so. I went on in this vein, and Ece nodded enthusiastically and said she got the idea.

4.4. Excursus, continued

I responded as follows to the comment at the end of §4.2.

Should I emphasize that was referring to the book by Morris Kline, not a book by (for example) Jacob Klein [36]?

The method given by Cardano, applied to

$$x^3 - 15x - 4 = 0,$$

will I believe give as a solution the sum of the cube roots of $2 + 11\mathrm{i}$ and $2 - 11\mathrm{i}$, where $\mathrm{i}^2 + 1 = 0$. The method doesn't tell us, however, that these cube roots are $2 + \mathrm{i}$ and $2 - \mathrm{i}$, so that 4 is a root of the cubic.

Are you suggesting that, 450 years before Cardano—who apparently thought he was publishing the first numerical solution of a cubic—, Khayyám already knew about such solutions?

Under *cubic function, Wikipedia* says,

> In the 11th century, the Persian poet-mathematician, Omar Khayyám (1048–1131), made significant progress in the theory of cubic equations. In an early paper he wrote regarding cubic equations, he discovered that a cubic equation can have more than one solution, that it cannot be solved using earlier compass and straightedge constructions, and found a geometric solution which could be used to get a numerical answer by consulting trigonometric tables. In his later work, the Treatise on Demonstration of Problems of Algebra, he

> wrote a complete classification of cubic equations with general geometric solutions found by means of intersecting conic sections.

The information about Khayyám's "early paper" seems to be second-hand; there is no direct reference to such a paper, but to a webpage* that also uses the Khayyám quote that I gave (p. 130):

> Another achievement in the algebra text is Khayyám's realisation that a cubic equation can have more than one solution. He demonstrated the existence of equations having two solutions, but unfortunately he does not appear to have found that a cubic can have three solutions. He did hope that "arithmetic solutions" might be found one day when he wrote:—
>
> > Perhaps someone else who comes after us may find it out in the case, when there are not only the first three classes of known powers, namely the number, the thing and the square.

I don't know how to read this as other than admission that somebody in future may succeed where Khayyám and others have failed.

Again, Kline said,

> Omar Khayyam believed [the cubic] could be solved only geometrically, by using conic sections . . .

Perhaps he meant to say, Khayyám believed the only geometric solution was by using conic sections (and not straightedge and compass alone).

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*`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Khayyam.html`

4.5. Tuesday, March 9

I wonder if it is a bad idea to read mathematicians like Euclid and Newton without reading mathematicians like Cardano. Euclid is a model of exposition. In reading him, you may not be sure where you are going, but at least you know how you got where you are. Newton follows this model, more or less. Cardano does not. And yet, as I just told a student, Cardano's work is the direct ancestor of the mathematics being taught in another course in our department: Galois theory.

Burhan and Fuad presented sections 6 and 7 of Chapter VI of Cardano's *Ars Magna.* This is where Cardano establishes the rule that—said Burhan—every Turkish student learns today as

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Cardano proves it geometrically, by dividing a line AC at B, drawing a square on AC, then erecting a cube on this square. The square is $ACEF$. Cardano draws a line perpendicular to AC at B and marks on it BD equal to BC; through D a line is drawn parallel to AC. So the square is divided into regions DA, DC, DE, and DF, and Cardano considers what happens when you multiply their sum by the sum of AB and BC.

Burhan worked out the product in detail, as Cardano does. I thought his classmates would get bored and start chatting amongst themselves; but they remained mostly quiet. When Burhan was finished, I asked if he was now more confident in the truth of the identity he had learned years ago; he said yes.

Again, Cardano writes out the cube of AC in terms of AB, BC, and the regions DA, DC, DE, and DF. So he gets volumes like $AB.DA$ and $BC.DF$, which he observes to be equal to $AB^2.BC$, and so on. That is, he introduces the new letters of the diagram, only to reduce everything in the end to AB and BC.

I went to the board and asked: We didn't Cardano just compute directly,

$$\begin{aligned}(AB+BC)^3 &= (AB+BC)(AB+BC)^2\\ &= (AB+BC)(AB^2+2AB.BC+BC^2)\end{aligned}$$

and so on? I think it was Fuad who said this wasn't acceptible, because it was not geometric; it was just symbol manipulation. He had earlier named "distributivity" as the algebraic rule at work in such an argument.

As Cardano does, Fuad then went to the board to derive the rule that

$$AB^3+3AB.BC^2$$

exceeds

$$BC^3+3(BC.AB^2)$$

by

$$(AB-BC)^3.$$

Today we can obtain this by replacing BC with $-BC$ in the earlier identity. But Fuad gave Cardano's geometric argument. Cardano takes G on AC so that $AG = BC$. Then we know

$$AB^3 = (AG+GB)^3 = AG^3+GB^3+3AG.GB^2+3GB.AG^2.$$

Now add $3AB.BC^2$ to both sides, and replace AG with BC, getting

$$\begin{aligned}&AB^3+3AB.BC^2\\ &\qquad = BC^3+GB^3+3BC.(GB^2+GB.BC+AB.BC).\end{aligned}$$

One must show that the parenthetical quantity, $GB^2 + GB.BC + AB.BC$, is equal to AB^2. Fuad drew a picture and tried to fit the pieces together to make the desired square; but he just couldn't do it. Şule seemed to explain adequately what to do; it involved orienting one of the rectangles vertically rather than horizontally; but Fuad couldn't see it. I tried to get Şule to go to the board; but Oguzhan was the one who did it.

I thought probably Fuad could see things if he worked by himself; being at the blackboard can gum up the thought process, as new teachers may soon learn.

Anyway, now we had Cardano's Corollary 1. There's a Corollary 2, which *does* seem to be, for Cardano, the result of replacing BC with $-BC$ (that is, minus-BC). Fuad hadn't really understood what Cardano was doing; I hadn't either, except in the way I just said. I don't really know what a negative number is for Cardano, if it is anything at all.

When I was reading Cardano early this morning, I thought: Homer continues to be great poetry; but Cardano does not continue to be great mathematics. If you want to see where our mathematics comes from, you must read people like Cardano; but otherwise there is no point.

Unless the point is that important mathematics need not be well written. Students of mathematics today should read Euclid as a model for the logical development of mathematics. But there is a claim that the best mathematicians do not write like Euclid: they are to busy proving things to polish their work. Maybe students should be aware of this too.

In class, it was break time. Again MuYaKu came up to ask me something. This time it was, Why didn't Cardano just draw the full cube? I thought somebody had raised this question in class. I suggested that printing the necessary diagram might have been too much of a challenge. But spatial intuition itself was probably not a barrier for Cardano. One thousand, eight hundred years before Cardano, Euclid had had an outstanding spacial intuition. To show this, I drew on the board the diagram that Euclid uses in the construction of the dodecahedron.

This was in the break; but then Seray asked if she and her partners Makbule and Salih (Acar) could make their presentation on Thursday. I said OK; how could I not? I did have some things to talk about.

Indeed, I started the next hour by talking about my diagram from Euclid. (For independent reasons, I had spent the weekend studying "polytopes": analogues of polygons and polyhedrons in higher dimensions. This is why Euclid's diagram was fresh in mind.)

Seray and her partners will present Cardano's solution of a cubic equation. Today then I reviewed Khayyam's solution. I *derived* Khayyam's solution to

$$x^3 + a^2 b = a^2 x$$

in the manner I suggested in my log entry written on March 3: rewrite as

$$x^3 = a^2(x - b), \qquad \frac{x^2}{a^2} = \frac{x-b}{x};$$

now introduce y so that

$$\frac{x}{a} = \frac{y}{x} = \frac{x-b}{y},$$

and find the intersection of the curves given by

$$\frac{x}{a} = \frac{y}{x}, \qquad \frac{y}{x} = \frac{x-b}{y},$$

that is,

$$ay = x^2 \qquad\qquad y^2 = x(x - b)$$

—a parabola and hyperbola, respectively.

As an **exercise,** I suggested solving all other cubics in this style: for example, cube and sides equal number—which I solved myself; the curves are a parabola and circle.

I recalled that MuYaKu doubted that such "solutions" were really solutions. But I repeated something Ali had presented last time. I wrote it thus: The equation

$$a_1x + a_3x^3 + a_5x^5 + \cdots + a_{2n+1}x^{2n+1} = b$$

(all numbers positive) definitely has a solution, for Cardano, presumably because, as x grows from 0 without bound, so does the left hand side. I drew a graph of this. (I also wrote the word "anachronistic" on the board, to make sure they knew the word I used to describe my algebraic treatment; nobody admitted to knowing the word when I said it out loud.)

But how do we know that the left hand side of the equation passes through every value? It seems to me that we can be more confident that Khayyám's solution of a cubic really *does* establish the existence of a solution. I said we could accept that the parabola and the hyperbola in one of Khayyam's solutions really did intersect. Actually Ali reminded me that they might *not* intersect, in certain cases of the parameters of the equation.

I ended class about five minutes early. Salih, Seray, and Makbule asked about the reading they were supposed to present; for example, what about these numbers 1570 and 1663 in the footnotes? I explained that those were dates of later editions of the *Ars Magma.* What did "binomium" and "apotome" mean?

I tried to give the Euclidean definition first, but I had only the Green Lion *Bones* summary [22], which doesn't have definitions; I forgot however that the terms are defined in *Propositions* X.36 and X.73. Anyway, I said that, for Cardano apparently, they are expressions like A plus or minus the square root of B. We had a bit more discussion; for example, Cardano solves the cubic

$$x^3 + 6x = 20.$$

One can see that this has the root $x = 2$. But Cardano's method (or perhaps rather Tartaglia's method) gives the difference of cube roots of a binomium and an apotome. Are they the same? Salih asked. What is the use of Cardano's complicated solution? I observed that one could compute that solution approximately, then show that it must be 2, though I didn't think there was an algorithm for general simplification of solutions. I mentioned the Galois theory course, as I said. I suggested that Salih and his partners could look up "cubic equation" on *Wikipedia* for ideas, if they wanted. Cardano himself is obscure.

4.6. Thursday, March 11

Salih, Seray, and Makbule presented Chapter XI of Cardano: "On the Cube and First Power Equal to a Number." It's about cubes, as the title says; but Cardano's diagram is of two-dimensional regions. Salih started class by trying to draw a real cube, divided into sections; but he couldn't get it right, so Seray came to do it. Then Salih proceeded with his demonstration, in which he claimed to show what I shall express algebraically as:

If $u^3 - v^3 = 20$, and $3uv = 6$, and $x = u - v$, then $x^3 + 6x = 20$.

Cardano's apparent purpose is to *solve* the equation $x^3 + 6x = 20$. So $u - v$ is a solution, except Cardano doesn't say till later how he gets u and v. For him they are lines AC and BC, with B lying between A and C. Salih just said, more or less with Cardano,

let $AC^3 - BC^3 = 20$, and $AC.BC = 2$.

How can we just let it be so? I asked. Salih apparently hadn't considered this question, because Cardano doesn't. Well then, Cardano is a bad expositor; I said this, and the students chuckled.

Seray took over at the point where Cardano says, "Now assume that BC is negative." She went through the calculations that Cardano apparently does, but she couldn't say clearly what the point was. I'm not sure what the point is either. Seray seemed to suggest that Salih did the positive case, and she the negative.

I think rather that Cardano just has a long-winded way of arguing that $AB^3 + 6AB = 20$; assuming BC is negative means *subtracting* BC from AC to get AB.

Makbule went to the board to work out Cardano's "rule" for finding a numerical solution to $x^3 + 6x = 20$: she went through the stated manipulations of 6 and 20 to obtain the solution

$$x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}.$$

She couldn't explain *why* x could be so found. Apparently she hadn't actually checked by substitution that this x worked. I think she agreed with me that this value of x must be equal to 2. But then Seray said it is *approximately* equal to 2. She must have misunderstood what I said to her and Salih at the end of the last class (and reported in my notes here).

I argued that the equation $x^3 + 6x = 20$ can have only one positive solution. Since 2 is obviously a solution, and the complicated thing above is a solution (Ali reported that it really was: he checked it), then they must be equal.

But then Ece said a cubic equation could have three solutions? Ali said there were *at most* three solutions, but some of them might be repeated. But Ece couldn't give a reason why there should be 3 solutions; she had just heard it somewhere.

Students seemed familiar with the idea of multiple roots. How many cube roots has 1? I asked. Just one, they said, but it was a multiple root. I showed this was wrong: $x^3 - 1$ factorizes as

$$(x-1)(x^2+x+1),$$

and we can solve $x^2 + x + 1 = 0$; the solutions are not $x = 1$. I try to stress that this was not the sort of quadratic equation that our writers had considered, since it has no positive root. However, Zhala knew that Cardano would be considering square roots of negative numbers in the next reading (which she and her friends will present).

Nobody admitted to knowing what Cardano was really doing in his solution of a cubic. I wrote on the board what I wrote above, but in more general terms:

If $u^3 - v^3 = b$, and $3uv = a$, and $x = u - v$, then
$$x^3 + ax = b.$$

I checked it by working out the cube of $u - v$. Meanwhile, I left the class with the **exercise:**

If $u^3 - v^3 = b$, and $3uv = a$, then what are u and v?

Cardano knows how to find u and v, since his "rule" requires it. But unless I am missing something, Cardano does not explain to the reader how his rule can be derived.

The derivation is pretty easy for us now: We get

$$u^6 + u^3v^3 = 6u^3, \qquad u^6 + 3a^3 = 6u^3,$$

and the latter equation is quadratic in u^3. But did Cardano know how to do this? He must have, in some sense. But in Chapter 1, when he looks at equations like $x^4 + 3x^2 = 28$, which are quadratic in x^2, he doesn't give an example that is quadratic in a cube.

Also, Cardano wasn't the first to solve the cubic. In the preface to his translation, Witmer writes:

> It was [Cardano] who developed the proof that the formula or formulae that he received from del Ferro and Tartaglia are correct; found the method for reducing the more complex forms of the cubic . . . to one or another of the simple forms . . .

I see no suggestion that Cardano could *derive* the formulas in question. Some scholarship is needed here, I think.

4.7. Tuesday, March 16

In today's two-hour class we discussed chapter XXXVII of Cardano's *Ars Magna,* called (in Witmer's translation) "On the Rule for Postulating a Negative." Yasemin, Zhala, and Duygu presented the material.

First we waited for Duygu to show up; Zhala called her. I asked the students why they thought I wanted them to make presentations. Melis said, "To give us experience in lecturing."

Ece said, “To make sure we follow the material” (or something to that effect). I agreed with such reasons, but said that I also wanted to learn from the students. Again I pointed out that Cardano’s book was not the Quran or the Bible; it was just written by some guy, who could make mistakes or be confusing.

Still, Yasemin began her presentation by reciting from memory the beginning of the chapter:

> The rule is threefold, for one either assumes a negative, or seeks a negative square root, or seeks what is not . . .

I said I hoped she would explain what this all meant.

As I see it, the gist of the chapter is this: There are some problems whose solutions are normally positive numbers; if you change the parameters, the solutions may become negative, but the same general method of solution works. In other cases, the solutions may involve square roots of negative numbers; such solutions may not make sense, but they still work in a “formal” sense. (Note that “formal” here does not refer, as it could, to the *highest* level of reality, but to one of the lowest.)

Cardano observes, for example, that the equation

$$x^2 + 4x = 32$$

has the solution 8, while the related equation

$$x^2 = 4x + 32$$

has the solution 4; this means the former equation has also the solution minus-4.

Now, I should think that Cardano would compute the solution to the first equation as

> half of 4 plus the square root of the sum of 32 and the square of half of 4.

He also knows that a number has two square roots, one being negative; why does he not then observe that, in the last computation, if the negative square root is used, one indeed gets minus-4? Why does he instead convert to the second equation above? Does he think this conversion makes the solution minus-4 more plausible?

Cardano illustrates with a word problem: Francis's wife's dowry is 100 gold pieces more than Francis's own wealth; and the square of the dowry is 400 more than the square of Francis's wealth. Cardano doesn't make the reasoning explicit, but it follows that Francis must be in debt: his wealth is "minus-x." Then one gets the equation

$$x^2 + 400 = (100 - x)^2,$$

which one solves to find $x = 48$; so Francis is 48 gold pieces in debt, and his wife's dowry is 52.

I had a lot to say about all of this, and the students had comments as well. Oguzhan asked why Cardano doesn't just let Francis's wealth be x; then we would just find $x = -48$. Zhala drew a vertical number line, with Francis's wealth below 0, and his wife's dowry above.

I observed: Cardano says at one point that the difference between the squares is 400 *gold pieces* (Witmer leaves Cardano's *aurei* untranslated). But the difference is 400 *squares* of pieces—which however has no physical meaning that I know of. I suggested that nobody would ever be interested in the situation of Cardano's problem.

Finally, in solving the equation above, I would proceed with something like

$$x^2 + 400 = 10000 - 200x + x^2,$$
$$200x = 9600,$$

$$x = 48.$$

This is what Yasemin wrote, except that, like Cardano, she wrote the middle line here as

$$9600 = 200x.$$

That's fine, but in my mind it involves an extra step, either to switch members of an equation, or to cancel a minus sign. For me, I suppose, an equation is a spatial entity, with definite left and right sides. For Cardano, perhaps the left and right are not so distinct, and he can interchange equations $A = B$ and $B = A$ as easily as he might interchange the two expressions CD and DC for the same line segment. (Again, Cardano doesn't actually *have* equations in our sense; he just writes in words, This equals that.)

I told Yasemin she should decide whether Cardano's next two examples were worth going through; she decided they weren't.

Zhala worked through Cardano's problem of dividing 10 into two parts whose product is 40. (This problem is not numbered; it is just the first illustration of "Rule 2.") Cardano says, It can't be done, but do it anyway; you get that the parts are

> 5 plus the square root of minus-15, and 5 minus the square root of minus-15.

He checks this by performing the multiplication in a box in the text: I reproduce it as follows, using ℞ for the "Rx" symbol that Cardano (or his printers in 1663) use for a square-root sign.

5. p̃. ℞ m̃.15
5. m̃. ℞ m̃.15
25 m̃. m̃.15. quad. est 40.

But he calls this "as subtle as it is useless."

Oguzhan drew my attention to Cardano's comment,

> Yet the nature of AD [a square] is not the same as that of 40 or of AB [a line] . . .

I don't know that it sheds any light on square roots of negative numbers. Cardano does go on to observe that, whereas minus-15 is 5 squared minus 40, one could try taking the sum of 5 squared and 40 instead. This doesn't give the right answer. It does however suggest that Cardano may in other cases *guess* solutions to problems, and then verify them by substitution, rather than actually deriving them. (I raised this issue in my last log entry.)

In problem 4, Cardano proposes to divide 6 into two pieces, the sum of whose squares is 50. Cardano gets that the pieces are 7 and minus-1. Zhala worked this out after the break, just following Cardano's recipe, which is: Take half of 6, and add or subtract the square root of the difference of the 25 from the square of half of 6:

$$\frac{6}{2}\ \tilde{\text{p}}.\ \text{R}\!\!\!\!\text{x}\,25\ \tilde{\text{m}}.\ \text{sq.}\ \frac{6}{2}, \qquad\qquad \frac{6}{2}\ \tilde{\text{m}}.\ \text{R}\!\!\!\!\text{x}\,25\ \tilde{\text{m}}.\ \text{sq.}\ \frac{6}{2}.$$

Fuad asked why this worked; Zhala didn't know. I pointed out that the recipe differs from the recipe for solving a quadratic equation: in the latter case, under the radical sign, the square of half the number of roots is always *added,* never subtracted.

I proposed the rule that I thought Cardano was following. Maybe he gives it earlier in the book, in a part we didn't read. I drew pictures for this rule, but algebraically it is:

$$(a-x)^2+(a+x)^2=2(a^2+x^2).$$

In our problem, $a = 6/2$, and 6 is divided unequally into pieces $a - x$ and $a + x$; the sum of their squares is 50, and therefore

$$a^2 + x^2 = 25,$$
$$x = \text{℞}(25\,\tilde{\mathrm{m}}.\,9) = 4;$$

so the pieces are $\tilde{\mathrm{m}}.1$ and 7.

Even if Cardano does work out this kind of solution earlier in the book, maybe it's better not to read the solution, but rather come up with it on one's own. But it took me a long time to find the solution myself.

Cardano's Rule 3 seems to be about numbers that involve both negatives and square roots of negatives. Duygu showed that Cardano's only example under this rule is simply wrong. Cardano seeks three numbers, which Duygu labelled I, II, and III. We want then

$$\frac{I}{II} = \frac{II}{III}$$

and also further conditions. (I thought it wonderful that Duygu used Roman numerals as variables.) If I is a square, as x^2, then the conditions are:

$$II = x^2 - x, \qquad III = x^2 - x - \text{℞}(x^2 - x).$$

Cardano asserts that I, II, III are

$$\frac{1}{4}, \qquad \tilde{\mathrm{m}}.\,\frac{1}{4}, \qquad \tilde{\mathrm{m}}.\,\frac{1}{4}\,\tilde{\mathrm{m}}.\,\text{℞}\,\tilde{\mathrm{m}}.\frac{1}{4}.$$

Indeed, Cardano takes the product of I and III, claiming it is

$$\tilde{\mathrm{m}}.\frac{1}{16}\,\tilde{\mathrm{p}}.\,\text{℞}\,\frac{1}{64},$$

which is 1/16, the square of II; but the product is really

$$\tilde{\text{m}}.\frac{1}{16}\,\tilde{\text{p}}.\,\text{R}\!\!\!\text{x}\,\tilde{\text{m}}.\frac{1}{64}.$$

Either Cardano forgot the minus-sign, or he is confused about its importance. I remarked that the translator didn't note a problem, although he had caught an error earlier on the page.

We had ten minutes left, but Şule wanted to start presenting Chapter XXXIX, section 3. She gave a preliminary algebraic result, presented geometrically, needed for solving quartic equations (equations involving squares of squares). She didn't make clear why the result was needed though, until I questioned her at the end.

4.8. Thursday, March 18

The preliminary result presented by Şule is needed in the form

$$(x^2+6)^2+2(x^2+6)t+t^2=(x^2+6+t)^2.$$

Before Şule made use of this today, I made assignments for our next readings, in Viète and Descartes. MuYaKu asked for one of these assignments, though I thought he was going to talk, with Şule, about Cardano; but he said he hadn't been able to understand Cardano.

The assigned readings were:

1. Chapters I–III of Viète's *Introduction to the Analytic Art* [36, Appendix], along with the fifth of the "laws of zetetics" in Ch. V.
2. Book I of Descartes's *Geometry* [18], divided into these sections:
   a) pp. 2–6,

b) 6–11,
c) 11–17,
d) 17–25,
e) 25–37.

Şule stated the problem that Cardano takes up: to find three numbers in proportion whose sum is 10 and the product of the first two of which is 6. If the middle number is x, then the first is $6/x$, and the third is $x^3/6$, so

$$\begin{gathered} \frac{6}{x} + x + \frac{x^3}{6} = 10, \\ 36 + 6x^2 + x^4 = 60x, \\ 36 + 12x^2 + x^4 = 6x^2 + 60x. \end{gathered}$$

The left-hand side is now a square, $(x^2+6)^2$. This is still a square if we add $2(x^2+6)t + t^2$ as above, getting

$$(x^2 + 6 + t)^2 = (2t+6)x^2 + 60x + t^2 + 12t. \tag{§}$$

The right-hand side is now a square if and only if

$$\begin{gathered} (2t+6)\cdot(t^2+12t) = \left(\frac{60}{2}\right)^2 = 900, \\ t^3 + 15t^2 + 36t = 450. \end{gathered} \tag{¶}$$

Şule did all this, following Witmer's translation pretty closely (she used Witmer's b instead of my t; again, Cardano himself uses no such letters). Then Şule didn't know what to do. I pointed out that if (¶) holds, then we can take square roots in (§), getting

$$x^2 + 6 + t = \sqrt{2t+6} + \frac{30}{\sqrt{2t+6}},$$

a quadratic in x.

From Cardano, we had learned to solve cubics only if there was no term in t^2. I left it as an **exercise** to eliminate this term from (¶) by letting t be s minus something.

Ali observed that we could solve quartics now only if there was no x^3 term; I left it as another **exercise** to eliminate such terms, when present.

Again, from the equation

$$x^4 + 12x^2 + 36 = 6x^2 + 60x,$$

Cardano derives

$$t^3 + 15t^2 + 36t = 450.$$

He then asserts a general rule, which, as Şule had apparently observed, was wrong: The coefficient of t^2 in the second equation is always five-fourths the coefficient of 12 in the first. Neither Witmer nor Struik appears to notice this mistake. It's hard to see why Cardano would make the mistake, unless one remembers that Cardano is reasoning with ordinary words, not our algebraic symbolism. (He also seems to be less advanced than Euclid in his concern for proof.) If we do use algebraic symbolism, then from

$$x^4 + 4ax^2 + 4a^2 = 2bx^2 + 4cx, \tag{$\|$}$$

that is,

$$(x^2 + 2a)^2 = 2bx^2 + 4cx,$$

we get

$$\begin{aligned}(x^2 + 2a + t)^2 &= 2bx^2 + 4cx + 2tx^2 + 4at + t^2\\ &= (2t + 2b)x^2 + 4cx + t^2 + 4at,\end{aligned}$$

and we require

$$\begin{aligned}2c^2 &= (2t+2b)(t^2+4at)\\ &= 2t^3+(8a+2b)t^2+4abt,\\ t^3+(4a+b)t^2+4abt &= 2c^2. \qquad (**)\end{aligned}$$

Cardano seems to have compared (‖) and (∗∗) in a special case and drawn the wrong conclusion.

After class, Gökçen asked me about higher dimensions. Now, Omar Khayyám had written,

> If the algebraist uses the "square-square" in problems of geometry it is only metaphorically, not properly, for it is impossible that the "square-square" be counted as a magnitude. [35, p. 557]

Why then was Gökçen's topology class (I think it was topology) talking about higher dimensions? I talked to her a bit about hypercubes, but then I had to run for the departmental seminar.

5. Viète and Descartes

5.1. Tuesday, March 23

To today's class though, I brought a (projection into three dimensions of a) hypercube, made with my old Ramagon™ pieces. I came early to class and found Şule and Mehmet Arif Şekercioğlu standing outside. Both of them were curious about the model I was carrying, but Mehmet was the one who took it from my hands. It was however Şule who recognized that the model indeed consisted of two connected cubes (just as a cube itself consists of two connected squares).

Then Gökçen came. In the classroom I talked more about the model and higher dimensions in general, pointing out for example that the vertices of the hypercube can be given, in $\mathbb{R}^4$, the 16 coordinates

$$(\pm 1, \pm 1, \pm 1, \pm 1),$$

while the 4 vertices of usual 3-dimensional tetrahedron can be given the coordinates

$$(1, 0, 0, 0), \qquad (0, 1, 0, 0), \qquad (0, 0, 1, 0), \qquad (0, 0, 0, 1).$$

Then I gave way to Melis and Ece for their presentation of the Viète reading.

In fact Ece and Melis had come to my office two hours before class, saying they had not picked up the reading till yesterday, and it was too long for two people. I showed them that it wasn't so long, and I talked to them about it. I discussed the Greek meanings of *ζητητική*, *ποριστική*, and *ῥητική*; I think I skipped *ἐξηγητική*, apparently used as a synonym for *ῥητική*. I mentioned that the earlier name of Beyoğlu in Istanbul, namely Pera (Πέρα), meant "beyond," that is, beyond the Golden Horn; this word was apparently related to *ποριστική*. I pointed out that, according to a note that wasn't in their Viète photocopy, the distinction between zetetic and poristic may have corresponded to the distinction between theorem and problem that we had discussed in Math 303. Melis recalled this, but Ece had not been in that class. In any case, I didn't claim to understand just what Viète meant by zetetic, poristic, and exegetic.

Nonetheless, in her presentation, Melis spoke as if she understood the distinctions between these words. On the board, she made a diagram: Figure 5.1. She said something about

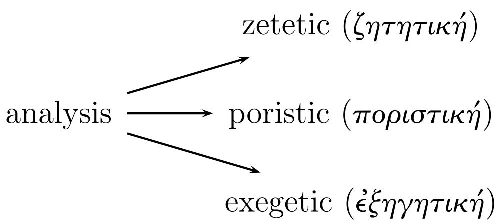

Figure 5.1. Viète's analysis of analysis

how zetetic involves *finding* a solution: for example,

$$x^2 + 21 = 10x, \qquad (*)$$

finding $x = 3$ or $x = -13$. But poristic for her seemed to involve a case where you just *see* a solution. The exegetic was where the solution just *comes out;* Melis here pointed to the *ex* part of the word. The presentation had something of the air of B.S.; but then, in the absence of examples, so does Viète's presentation.

Melis had begun by writing *"The Analytic Art"* on the board; I pointed out that we were reading only the *Introduction* to this, although I had asked the library to order the whole thing [56].

Finishing with Chapter I, Melis made an obscure reference to Viète's vague comment about working not with numbers, but with "species." I indicated the equation ($*$) that she had written, asking whether, according to Viète, we were not going to work with such equations. Melis didn't have much to say, so I mentioned the footnote referring to the theory that "species" meant letter, as A, B, or C.

Melis proceeded to Chapter II, which lists the "stipulations"; Melis provided the Turkish translation *şart.* Melis read out the stipulations, writing out their bracketed symbolic translations in the text. I wondered what she thought the value of this was, since she wasn't really trying to *explain* anything. She mentioned at the beginning that the first stipulations were Euclid's Common Notions; she agreed when I said the later stipulations weren't common notions for Euclid. I suggested that the common notion

Equals to the same are equal to each other

is a lot different from a claim like Viète's 7th stipulation,

$$\text{if } a : b :: c : d, \text{ then } a : c :: b : d.$$

Indeed, the latter doesn't make sense, in Euclid's terms, if, for example, a and b are triangles in the same parallels, and c and d are their bases: In Figure 5.2 we have

$$ABD : BCD :: AB : BC,$$

but ABD does not even *have* a ratio to AB,—much less does it have the *same* ratio that BCD has to BC. (I noticed that Oğuzhan—who always sits in front—was writing this down.)

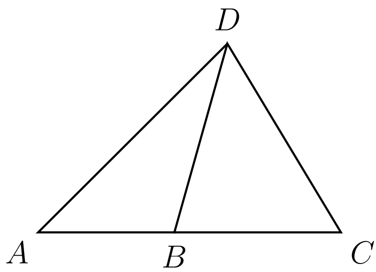

Figure 5.2. Ratios in triangles

But by Viète's last stipulations, the proportion

$$a : b :: c : d$$

is equivalent to the equation

$$ad = bc.$$

I asked what this meant applied to Euclid's XII.18,

> Spheres are to one another in the triplicate ratio of their diameters.

If the spheres are S and s, and the diameters D and d, then

$$S : s :: D^3 : d^3;$$

for Viète then,

$$Sd^3 = sD^3;$$

but what does this *mean?*

Ece presented Chapter III, on the "law of homogeneity." She wrote a couple of Viète's statements as equations:

$$\text{h} + \text{h} = \text{homog.}$$
$$\text{h} \cdot \text{h} = \text{heterog.}$$

She admitted in the break that she wasn't sure what the first one meant, and indeed Viète's statement,

> if a magnitude is added to a magnitude, it is homogeneous with it,

has pronouns with uncertain antecedents. But probably Viète means that two magnitudes cannot be added unless they are homogeneous.

Before the break, Ece had written down Viète's "ladder-rungs"—in Turkish, *merdiven basamakları.* As Melis had written down Greek forms, so Ece wrote down the Latin:

1. side (*latus*) or root (*radix*),
2. square (*quadratum*),
3. cube (*cubus*),
4. squared square (*quadrato-quadratum*)

—I think Ece stopped there, fortunately, without trying to write down Viète's whole list up to the cubed-cubed-cube. I asked what a squared square was; Ece *didn't* say that my hypercube was one, so I did, while admitting I had no evidence that Viète thought in such terms.

After the break, Ece continued with the "genera of the compared magnitudes":

1. length (*longitudo*) or breadth (*latitudo*),
2. plane (*planum*),
3. solid (*solidum*),
4. plane-plane (*plano-planum*)

—again she stopped here, without going up to the solid-solid-solid. I quizzed her about the word *genera,* getting her to admit that it was the plural of *genus.* I led her to say that *homogeneous* meant having the same genus; I'm not sure she had fully recognized this. She gave some examples of "conjoined powers" from the italicized text, which had apparently been added by an editor. She quoted the given rule about how many conjoined powers there are at a given rung; but she gave no sign of having understood it. (I don't understand it myself.)

During the break, Salih Kanlıdağ—who with MuYaKu would be on the second team of presenters of Descartes—asked if he could present, not the coming Thursday, but the one after that, since the Exam would be next Tuesday, and he had another exam as well.

"Have you *read* the Descartes?" I asked.

He hadn't; I said I didn't think it would be a problem to prepare for Thursday, so he should at least try.

As it was, Mehmet Doğan and Gökçen finished with their assignment in Descartes. Mehmet made Descartes's argument that lines could be multiplied and divided to produce lines. He said that for Descartes,

$$a : b :: c : d \tag{†}$$

meant the same thing as

$$ad = bc.$$

Hadn't Viète already said that? I asked. Mehmet claimed that, for Viète, the proportion and equation were merely *equivalent,* not identical. But he also said that the notation in (†) was merely the convention of Descartes; I said I hadn't recalled seeing it in Descartes; in the passage in question, on the first page, Descartes just wrote out the proportion in words.

I said that I had recently published a paper [41] of new results that had been inspired by Descartes's figure (Figure 5.3), in which $AB : BD :: BC : BE$ because $DE \parallel AC$.

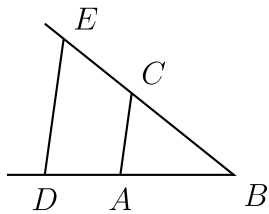

Figure 5.3. Descartes's geometric arithmetic

Gökçen talked about Descartes's formulation of what was in effect the law of homogeneity. She had asked me about this during the break: I don't recall exactly what her question was, but I observed that 1 cm^2 plus 1 cm was not really 2 of anything in particular; for then it should also be 100 mm^2 plus 10 mm, or 110 of something. Gökçen didn't repeat this example in her presentation, but she said you couldn't take $a^2 + a$ unless you had a designated unit, as b; then you could take $a^2 + ab$.

After class, Oğuzhan asked me about Descartes's own example: in taking the cube root of $a^2b^2 - b$, one should think of this as

$$\frac{a^2b^2}{u} - bu^2,$$

where u is the unit. Why, Oğuzhan wondered, did Descartes convert everything to a solid? Why not a plane, say? I think I suggested that he could have converted to $a^2b^2 - bu^3$, but perhaps he still had a prejudice against powers higher than 3. But now I see that I missed something: Descartes wanted to take a *cube* root, and that's why he wanted the radicand to be a solid. If the class were really a *discussion,* and students asked their questions to the whole class, rather than to me, perhaps somebody might have pointed this out.

5.2. Thursday, March 25

Today I asked Oğuzhan about his question from last time, and he said he had later understood the importance of Descartes's taking the *cube* root. Somebody wanted me to review for the exam. I just mentioned that we had read al-Khwārizmī, Thābit ibn Qurra, Khayyām, and Cardano; students should know how to solve problems in their styles. I did make sure that somebody could do the exercise from §4.6, p. 145. I quickly repeated the Khayyām-style solution of

$$x^3 + a^2x = a^2b,$$

noting the use of the Law of Homogeneity. Then I noted that Cardano's method is somewhat neater when applied to

$$x^3 + 3a^2x = 2a^2b,$$

since here if we let $x = u - v$, we get

$$u^3 = a^2(\sqrt{a^2 + b^2} + b),$$
$$v^3 = a^2(\sqrt{a^2 + b^2} - b)$$

(I'm not positive I didn't write u and v instead of u^3 and v^3).

Salih and MuYaKu made their presentation of pp. 6–11 of Descartes, though now, five days later, I can't remember just how they divided this section up. There wasn't much to say, though Salih did present the content of footnote 12 on p. 9 (which I hadn't read).

Mehmet Arif Şekercioğlu had stopped by my office, perhaps the previous day, to ask about his assignment, pp. 17–25, on the locus problems. He didn't have a partner; but Mihail, who had joined the class late, had been absent when I took students for Descartes; so I told Mehmet that he could work with Mihail. Mehmet was worried, though, that his English wasn't good, and Mihail didn't speak Turkish. I said I thought Mihail *did* speak Turkish. (Mihail and his twin brother are from Turkmenistan, where they attended a Turkish-language school—I recall learning this when they took a first-year course with me.)

Ece came later to my office, also trying to figure out the reading assigned to Mehmet. Of course everybody should read everything, but it was good to see Ece taking this seriously.

In class then, I stated a proposition derived from Taliaferro's appendix to his translation of Apollonius [4]. Proposition III.54 has the result,

$$\frac{AF \cdot CG}{AC^2} = \frac{EB^2}{BD^2} \cdot \frac{AD \cdot DC}{AE \cdot EC}.$$

If we draw through the arbitrary point H on the conic section the straight line parallel to AC meeting AD at Y and DC at Z, and the straight line parallel to DE meeting AC at X, then, as an **exercise,** one can show

$$\frac{HX^2}{HZ \cdot HY} = \frac{EB^2}{BD^2} \cdot \frac{DE^2}{AE \cdot EC}.$$

Thus a conic section is a solution to a three-line locus problem.

5.3. Tuesday, March 30

Class was occupied with an exam. Two hours before the exam, Melis came to my office to say she had a migraine, but didn't have a medical report; could she take a make-up? I said we would work something out. Aside from Melis the two students who have never come to class—Tolga and Anıl—only Yasemin didn't come to the exam. So 19 students took it.

During the exam, three people asked about the first question:

> A straight line is cut into equal and unequal segments. What is the relationship between the square on the half and the rectangle contained by the unequal segments?

They didn't understand what it meant for lines to *contain* a rectangle. This was dismaying. But as last semester, so this semester, students ended up doing better on the exam than I expected. This time I think their skills at memorizing formulas helped them.

5.4. Thursday, April 1

Mehmet Arif Şekercioğlu and Mihail talked about their few pages of Descartes (17–25). In stating the three-line locus problem, Mehmet seemed to think that the angles to each of the three lines should all be the same, although they need not be right. I suggested that the angles could differ, but he didn't agree. I let it go.

In describing the five-line locus problem, Mehmet suggested that, if the distances are a, b, c, d, and e, then the fraction abc/de should be a given constant. I said there should be something else in the denominator, to satisfy the Law of Homogeneity. I think Şule had already tried to say this, in Turkish, but Mehmet didn't seem to get the point. Oğuzhan suggested that, if we had a unit as Descartes does, then we wouldn't need to worry about the Law of Homogeneity.

There were a few minutes left, but Mihail said he needed only a few minutes. He mentioned Descartes's claim that, with six to nine lines, the curve could be found by conic sections. I emphasized that this didn't mean the curve *was* a conic section, only that particular points could be plotted by means of conic sections. Here I briefly previewed the first part of Book II, pp. 40–55, which I had asked Ali and Emir to talk about next week, since no volunteers had been forthcoming.

Duygu, Yasemin, and Zhala were supposed to present the last part of Book I in the next class. After today's class though, Duygu and Yasemin asked to postpone their presentation, because they had an exam coming up. Zhala was not around. I don't think they had read their section anyway. They seemed to think Ali and Emir could skip ahead to Book II, with the trio then going back to Book I on Thursday. I pointed out that we had two hours of class on Tuesday. Eventually I said *I* would take their section, if they would be the first to present from Newton.

The exam had asked:

> A cube and nine sides are equal to ten. Find the side numerically, as the difference of the cube roots of a *binomium* and an *apotome,* by Cardano's method (really Tartaglia's method); your steps should be clearly justifiable.

Many students just used Cardano's formula without justification; I gave them three out of 5 points. Ali was one of those students, and he wasn't too happy about it. We talked on

Friday afternoon (April 2), when Ali seemed to be suggesting that the formula could be understood as obvious. He said something about sixth powers that I didn't understnd; but I had to cut him off in order to go to the algebra seminar.

Ali then sent me an email, with a new-to-me derivation of the formula. We are solving

$$x^3 + ax = b.$$

Letting $x = u - v$, we get

$$x^3 = u^3 - v^3 - 3uvx,$$

so

$$u^3 - v^3 = b, \qquad\qquad 3uv = a.$$

This is standard. But then Ali observes:

$$\left(\frac{u^3 + v^3}{2}\right)^2 = \left(\frac{u^3 - v^3}{2}\right)^2 + (uv)^3 = \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3,$$

$$u^3 = \frac{u^3 + v^3}{2} + \frac{u^3 - v^3}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} + \frac{b}{2},$$

$$v^3 = \frac{u^3 + v^3}{2} - \frac{u^3 - v^3}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} - \frac{b}{2},$$

so we easily get Cardano's formula,

$$x = \sqrt[3]{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} + \frac{b}{2}} - \sqrt[3]{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} - \frac{b}{2}}.$$

I wrote him:

So you are giving an alternative method for solving the simultaneous equations

$$uv = a/3, \qquad\qquad u^3 - v^3 = b.$$

Instead of finding $v = a/3u$ and substituting in the other equation, you find $u^3 + v^3$ and *then* get u^3 and v^3 by adding and subtracting.

I don't know that your alternative is shorter to write down, but it is more elegant, and by knowing it, one may more easily memorize the formula for x. Is this your point? (I'm at home and do not have your paper here.)

Do you think Cardano found the solution by your method? Myself, I don't think I have really understood how Cardano thought about solving equations. When he was just working by himself, did he use pen and paper? Did he use anything like our modern (Cartesian?) notation?

In Chapter XI, Cardano solves the equation $x^3 + ax = b$ in case $a = 6$ and $b = 20$. He spends a long time proving what in our notation is expressed by:

If $u^3 - v^3 = b$, and $uv = a/3$,

$$\text{then } (u - v)^3 + a(u - v) = b. \quad (*)$$

He doesn't appear to say *why* we should let u and v be so. He doesn't say,

> If we let $x = u - v$, then $x^3 = u^3 - v^3 - 3uvx$.

But was he *thinking* of something like this? If so, *why* would one think to let $x = u - v$?

After Cardano establishes his version of $(*)$, he immediately says, in effect,

> x is the difference of the cube roots of the binomial $[(a/3)^3 + (b/2)^2]^{1/2} + b/2$ and the apotome $[(a/3)^3 + (b/2)^2]^{1/2} - b/2$.

How does he know this? Would we understand this better if we too, like Cardano aparently, had read Book X of Euclid's Elements, where the terms "binomial" and "apotome" are defined and used?

I'm not sure that scholars have considered these questions! The passage of Cardano appears in both *A Source Book in Mathematics* by David Eugene Smith and *A Source Book in Mathematics [1200–1800]* by D.E. Struik. In footnotes, Smith translates Cardano's words into modern notation, but gives no explanation. Struik does provide "explanation" in that he solves $u^3 - v^3 = b$ and $uv = a/3$ by finding $v = a/3u$ and substituting. He doesn't address the question of why there is no indication of such a solution in Cardano. Do you have any ideas?

Ali did write back.

5.5. Tuesday, March 6

I signed people up individually for the first ten lemmas in Newton's *Principia.* We should talk about the definitions and the axioms somehow, but I don't know how.

I talked about the last part of Book I of Descartes, on the n-line locus problems, using the notes I had prepared some time before. Descartes goes to excessive length to prove by example that every line in the problem involves a new distance of the form $ax + by + c$. I just isolated one case to serve for all. I could see that most people's minds were elsewhere.

Indeed, it may be hard to get excited about these locus problems. Anyway, it was break time.

I had also looked up the quadratrix and the conchoid in [52] (the index is in [53]), since Descartes mentions them. Ali had actually looked it up on *Wikipedia,* but not found a picture. So with the start of the next hour, I described the curve:

If $ABCD$ is a square as in Figure 5.4, let DC be moved to AB at the same time that AD moves to AB. The intersection

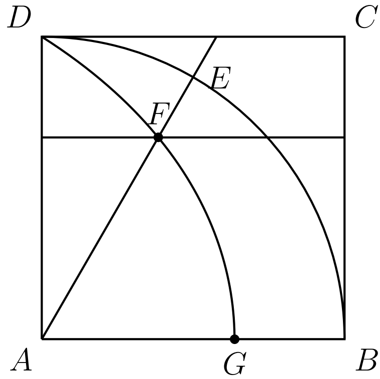

Figure 5.4. The quadratrix

of the two lines in motion, as E, traces out the quadratrix, DEG. I left it as an **exercise** to show:

$$DEB : AD :: AD : AF.$$

(Here DEF is a circular arc.) I said this curve allowed the squaring of the circle; indeed, the word *quadratrix* translates the Greek "tetragonizer"—that's how, on the spot, I translated the Greek *τετραγωνίζουσα*; I hadn't looked at the word before. Ali asked about this, so I just wrote it out: Find a straight line AH such that $AH : AD :: AD : AF$; then $AH = DEB$, so the triangle with base AH and height AD is equal to the circular quadrant $ABED$.

Emir had said at the end of the break that he couldn't understand the Descartes. I asked if he had talked to Ali, his supposed partner. He hadn't. They started talking right there. But this didn't do much good.

Actually, Emir started his presentation by writing a table on the board, listing the three kinds of problems:

(1) plane,
(2) solid,
(3) linear.

I asked what "linear" meant. Ali understood that this meant problems solved by lines in the sense of *curves.* Under *linear,* Emir wrote the quadratix and some other things. But then Emir just started *reading* Descartes out loud. Eventually I stopped him and asked what the point was. Then Ali stepped in and said some things. He didn't understand why Descartes should exclude the quadratrix but not other curves. I suggested we look at some other curves. Emir agreed to draw Descartes's funny contraption on p. 46. He worked out the equations for the curves drawn by the contraption.

Meanwhile, I noticed that nobody was paying attention. So when Ali offered to follow Emir, I suggested instead that we just quit. But first I had words with the students. I said I had hoped to break the model of education whereby the students face one direction, the teacher another. I said I wanted to learn from the students. Ali said "asymmetrical education" was the model everywhere in the world.

After everybody else left, Ali, Oğuzhan, and Besmir stayed behind. I asked if English was a barrier to classroom participation; they said No. I mentioned my general concern that math students took too many math courses, and that all students entered not just a university, but a *department,* when they couldn't really have a good idea what they were getting in for. Oğuzhan is in electrical engineering, actually; he said that's what he wanted to do, but he didn't really know what it meant before he came to METU.

Besmir just wanted to see his exam paper in my office. We went there *via* the library, so I could drop off the Newton printouts for photocopying. Along the way, Besmir asked about what I had wanted on Problem 4 of the exam: the solution of a cubic equation. I said I wanted a self-justifying solution. For example, if you solve a quadratic equation by the quadratic formula, this is not strictly self-justifying, although I could accept the formula as common knowledge. But if one really wanted a self-justifying solution of a quadratic, one would complete the square.

"What is completing the square?" asked Besmir.

I think I eventually got the point across. I also talked about Newton.

That night I wondered whether to cancel the attendence requirement, offering the students the option of basing their grade solely on exams.

6. Newton

6.1. Thursday, April 8

I gave up that idea. Burhan came by Thursday morning (I think it was then) to see his exam paper, and to make sure that his Newton assignment (Lemma 6) was really as short as he thought. I said Yes, but he should read everything else too.

Ece came by to ask about her assignment, Lemma 1, which didn't make sense to her. I said it wouldn't make much sense until we read what was to be done with the lemma. But I also handed her Dana Densmore's book [17], open to the relevant section, which Ece sat and read.

In class, I got everybody to sit in a rough semicircle, though there had to be empty desks in the middle; they could not all be pushed aside.

Ali announced that he had found a formula for the quadratrix, and asked if he could write it down. Of course, I said. I recall that his use of letters wasn't so clear, but he did use the tangent function. I suggested that, for Descartes, only curves given by polynomials were "geometrical," though I don't think he has a way of saying this.

Following Descartes, Ali derived an equation for the device on p. 50 (or p. 320 of the French original). But Ali didn't know why the equation defined a hyperbola in particular. Apparently he hadn't read the footnote on p. 55 giving Van

Schooten's argument.

I gave my own argument, with an additional line: GM, parallel to DF, as in Figure 6.1. Then

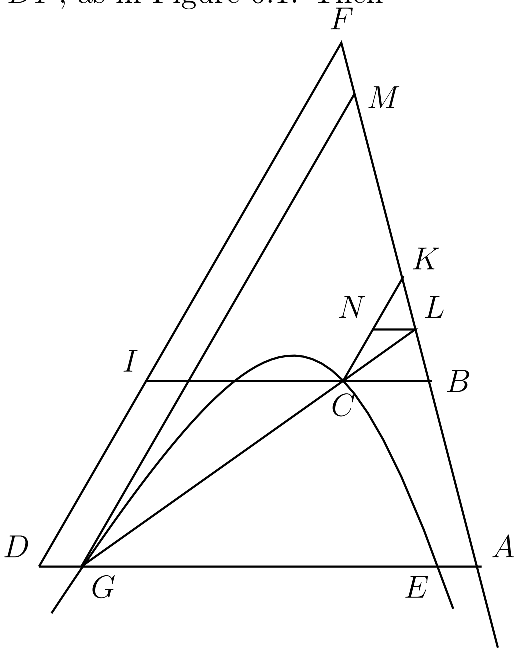

Figure 6.1. Descartes's construction of an hyperbola

$$ML : KL :: GL : CL :: GA : CB;$$

but since $DG = EA = NL$, we have also $GA = DE$ and hence

$$FM : DG :: KL : NL :: KL : DG,$$
$$FM = KL,$$
$$ML = FK,$$
$$FK : KL :: DE : CB,$$
$$IC : NL :: DE : CB,$$

$$IC : EA :: DE : CB,$$
$$IC \cdot CB = DE \cdot EA,$$

which is a condition that C is on a hyperbola with asymptotes FD and FA. I have just copied the argument from my notes; this is what I did in class, though I didn't always stop to follow the steps. Ali seemed to follow. Still, I claimed this argument was more faithful to the *picture.* Descartes gave us a way to just work out formulas without really thinking.

Ece wanted to talk about Newton's Lemma 1, since she would be away the following week like Duygu. (They will be fencing in Balıkesir, it seems.) She drew a stream of dots approaching another dot. I suggested that she was just proving that if a limit was approached, then the limit was reached (or something like that).

6.2. Tuesday, April 13

Today, Mehmet Doğan presented Lemma II; Besmir, III; Oğuzhan, IV; Şule, V; Burhan, VI; Yasemin, VII.

By the way, Mehmet had a facsimile of one of the old printings, not the *Wikipedia* transcription that I put in the library. He didn't seem to understand well what was going on. The key to Lemma I, I think, is the observation that the several rectangles of which the curve forms a sort of diagonal—these rectangles add up to the tallest of the circumscribed rectangles. But when questioned, Mehmet first denied that this rectangle vanished. Oğuzhan went to the board to explain.

Besmir argued out Lemma III orally, but did not make a clear picture; he just added a line to Mehmet's diagram, as in Newton. When questioned by me, Besmir made a picture

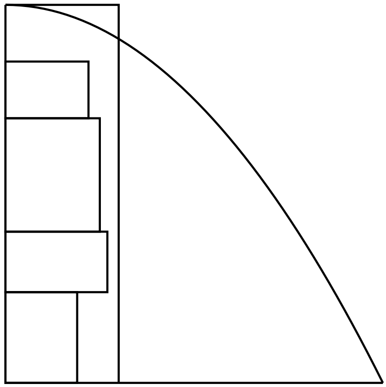

Figure 6.2. Unclear quadrature

something in Figure 6.2—mine is not a faithful reproduction, but the point is that Besmir did not make it clear where the stacked-up rectangles came from. If he understood, why didn't he make it clear in the picture? I got up and drew something like Figure 6.3.

Oğuzhan wasn't so clear on Lemma IV either, but then neither is Newton. The two figures look the same in the published diagram, when they probably should be as in Figure 6.4. Oğuzhan seemed to say that the bases of the figures were divided into proportional segments: If the one base is partitioned by A_0, A_1, A_2, etc., and the other by B_0, B_1, B_2, etc., then

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \alpha, \tag{$*$}$$

a constant, he said. I drew something like Figure 6.4 and wrote that the ratios $AB : A'B'$, $CD : C'D'$, $EF : E'F'$, etc., were assumed to be *ultimately* the same, and the conclusion was that $AGH : A'G'H'$ was this ratio. Oğuzhan said that's what

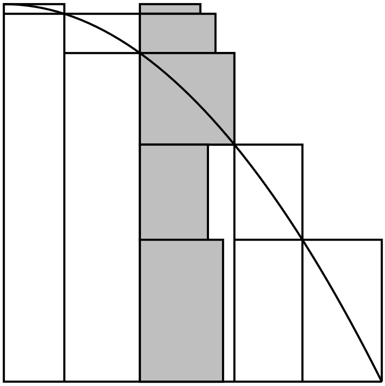

Figure 6.3. Newton's quadrature

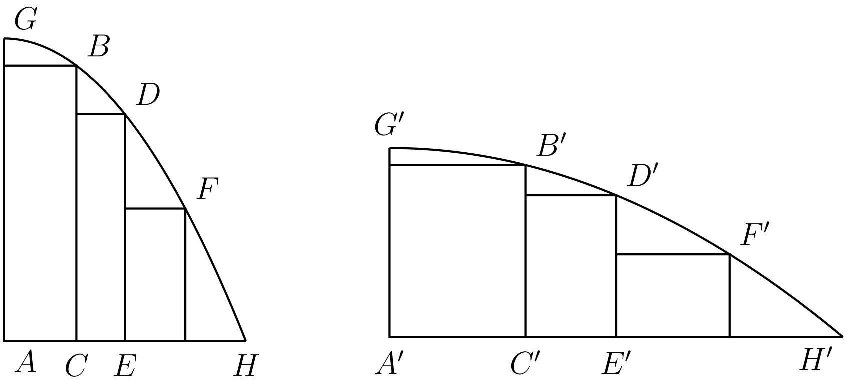

Figure 6.4. Proportional areas

he meant; he amended (∗) to something like

$$\frac{r(A_n - A_{n-1})}{r(B_n - B_{n-1})} = \alpha,$$

with r for rectangle. Anyway, Newton's hypothesis is not that the ratios $AB : A'B'$ are equal, but that they are *ultimately* equal; but what can this mean when a given rectangle doesn't obviously persist through the process of adding more rectangles?

We took a break. I think Burhan and Yasemin both asked me questions about their lemmas. I refrained from scolding them about not having asked me sooner. Oğuzhan said there was nothing new in the corollary to Lemma IV, so we proceeded to Şule. She seemed to suggest that Lemma V followed directly from IV. I drew two similar rectilinear (but non-convex) figures, saying that Euclid had shown them to be in the duplicate ratio of their sides; now Newton was saying the same was true even if there were curving sides. (I didn't quiz Şule on her understanding of *duplicate ratio.* She had written things like $|AB| = k|CD|$ to indicate ratio.)

I don't think Burhan got the point of Lemma VI. Perhaps he didn't understand the clause,

> the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle.

I tried to get him to draw this absurd situation, but he couldn't. So I did it. Meanwhile he had quoted something from Math 371, Differential Geometry, about curvature. I said we didn't have that knowledge.

Yasemin's Lemma VII gave us something to think about. She drew a diagram, but didn't write anything else till I asked her to. Then she wrote (for Corollary 2) that AD, DE, BF, FG, AB, and the arc ACB were ultimately equal (or "had the ratio of equality" as she kept saying, just parroting Newton). That is, she confused ED for AD (and FG for GB).

I said I didn't believe it. She said confidently, "It's hard to believe, but true!" I asked for a proof, but this was not forthcoming, just a remark that Newton was smarter than she (Yasemin) was. I asked if Yasemin's text had DE or AD. She checked, then corrected her statement. (Maybe that's when she said Newton was smarter than she was.)

Yasemin had drawn Newton's diagram, but apparently hadn't really seen the purpose of points b and d. All of the lines with capital-letter endpoints are vanishing; how can we talk about the ratios with which they vanish? I went up to make the argument. I also drew a new line be parallel to BE. I wrote something like:

$$Ab = Ad \text{ ultimately;}$$
$$Ab : Ad :: AB : AD \text{ now; therefore}$$
$$AB = AD \text{ ultimately.}$$
$$\text{Similarly, } AB = AE.$$
$$\text{Thefore } ED : AD \text{ is ultimately zero.}$$

There was some discussion of this. Oğuzhan didn't believe it at first. I had been saying things like "ED vanishes more quickly than AD," which he apparently thought meant ED got to zero *first.* Then he figured it out. Meanwhile Şule seemed to think Lemma VII followed immediately from Lemma VI.

It is good that we are getting into propositions that are attractive like puzzles. Before Lemma VII, I had wondered if we shouldn't have just jumped ahead to what Newton labels as Propositions.

6.3. Thursday, April 15

Salih Kanlıdağ presented Lemma VIII. He was as vague as Newton about what happens to the "distant points" b, d, and r. He seemed to think that, as B approaches A, so does R; I didn't think that was necessary.

Makbule had visited my office a few hours before. She had been absent on Tuesday, because of an exam in another course. She was supposed to present Lemma IX today, but didn't know if she could. I told her to work on it, visiting me if she had questions. She did visit later, and I discussed the lemma with her. I sketched some figures. She took the paper away with her. In class, she took that paper to the board with her.

She didn't draw Newton's figure right though: she didn't make B and b collinear with A. Salih Acar went to the board to straighten things out.

Duygu was away, as she had warned; but she was supposed to present Lemma X. I figured we could skip it for now, since I was keen to see how Mihail would do with Lemma XI. He was fine, but neither he nor anybody else seemed to know about the osculating circle. Many students had taken Math 371, Differential Geometry, and they could state that the "radius of curvature" is the inverse of the curvature; but they didn't understand the radius geometrically. I pointed out that Newton's AJ is the diameter of the circle of curvature.

6.4. Tuesday, April 20

Duygu showed up, but she thought she was supposed to present Lemma IX (rather than X). I told her she was wrong. Otherwise, the schedule was this:

1. Proposition I: Salih Acar;
2. Corollary I: Seray;
3. Corollary II: Zhala;
4. Corollary III: Melis;
5. Corollary IV: Mehmet.

However, Salih told me at the beginning of class that Seray had been in a car accident and would not be coming to class. It didn't sound as if Seray was seriously hurt. Was Salih prepared to take Seray's part? No, he had just found out she wasn't coming.

Otherwise, everybody presented their part; but all I really remember (writing eight days later) is that Mehmet said his corollary was immediate, and I accepted this.

I argued that Corollary II should be true by definition of center of forces, or by the second law of motion. Indeed, since arcs AB and BC are traversed in equal times, the chords AB and BC can stand for the average motions between A and B, and B and C, respectively. The change in the average motions is there represented by cC. Since this change is effected by forces directed towards S, *ultimately* cC must point towards S.

An argument is made in notes by Robert Bart [10, pp. 46–47] that I disputed as a student and still dispute. I offered it to my own students: Triangles SAB and SBC are ultimately equal (because the corresponding sectors of the orbit *are* equal). Therefore triangles SBC and SBc are ultimately equal. "Therefore by Euclid I, 40" [a supposed interpolation, according to which "equal triangles on equal bases and on the same side are in the same parallels"] Cc is ultimately parallel to SB.

But this is bad mathematics: We could have B, C, and c collinear, while maintaining the hypothesis that triangles SBC

and SBc are ultimately equal. I think the attempt here to fit Newton into the Euclidean mold is wrong-headed. I told all this to the students, as a warning not to trust commentators too much.

I also mentioned non-standard analysis [44], though I am not sure how to use it for Newton. Newton just assumes that centripetal force can be treated as acting discretely. In Abraham Robinson's terms, I suppose this amounts to partitioning time into intervals of infinitesimal length.

Since Friday was a holiday, and some students were going away Thursday, and I knew some other teachers cancelled their Thursday classes, I made my own class that day "optional."

6.5. Thursday, April 22

Three students came to the optional class: Oğuzhan and Besmir on purpose, MuYaKu by accident (he hadn't come Tuesday, and didn't know today's class was optional). When I asked how they liked Newton, all I could get was Oğuzhan's exclamation, "Amazing!" I offered them a precise definition of **ultimate equality** of ratios: A is to B ultimately as C is to D, provided that, for all k and m, we can take the magnitudes far enough (to their ultimate destination) that, assuming $kA \neq mB$ and $kC \neq mD$,

$$kA < mB \iff kC < mD.$$

6.6. Tuesday, April 27

I asked Duygu finally to present Lemma X, and she said she didn't know she was supposed to present it *today!* But she said she had read it a couple of times. She agreed to try to go through it anyway at the board, and she did pretty well. However, where Newton speaks of the "spaces" described by a body, Duygu thought he was referring to the "areas described by radii" of Proposition I. But she seemed to understand that Newton is obtaining distance by (what we call) integrating speed with respect to time.

Duygu later complained about the difficulty of the translation. I didn't keep a copy of the Motte translation that I made available to them, so I've been reading either Donahue [17] or Cohen and Whitman [40].

Burhan did Proposition II. I think he was confused by the language:

> And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBG, SCD, &c. about the immovable point S, (by prop. 40. book 1. elem. and law 2.) acts in the place B, according to the direction of a line parallel to cC . . . *

Burhan talked as if cC is *given* as parallel to AB, and that the equality of SBC and SBc follows. I said that the converse was what was to be proved, and he claimed to understand, but I had trouble being sure.

For ambiguity, compare Euclid's *Elements,* Propositions 18 and 19, in Heath's translation:

> In any triangle the greater side subtends the greater angle.
>
> In any triangle the greater angle is subtended by the greater side.

*The last line was aC in the Wikisource text, but I have now made the correction.

These are two parts of a biconditional; but which part is which?

Oğuzhan presented Proposition 3, the point of which seems to be that, in studying the earth–moon system, we can ignore the influence of the sun. Indeed, somebody—Oğuzhan or Ali, I think—already knew that Newton's L could stand for *Luna,* and T for *Terra.*

Zhala got started with Proposition 4. She had earlier visited my office, so I expected her to be comfortable with proving the main proposition. But in class she seemed either to consider the proposition obvious, or to believe that it could be derived from the corollaries. In any case, she stated the main proposition without any proof that I could recognize, and then she proceeded to the corollaries. I objected, but soon we ran out of time.

6.7. Thursday, April 29

We spent the whole time with Zhala's presentation of Proposition 4, but didn't quite finish. I think she was better prepared to prove the main theorem. Still she was slow, and her notation was confusing. She would write things like

$$\frac{f_1}{f_2} \propto \frac{{\ell_1}^2/r_1}{{\ell_2}^2/r_2}$$

I didn't notice this at first; later I said she should write $=$ instead of $\propto$, or else write

$$f \propto \frac{\ell}{r}.$$

She preferred to continue to work explicitly with ratios.

I recall being at the board for Corollary 3: period is constant if and only if force varies as the radius, $F \propto R$. I asked if this sort of situation actually happened in nature. I think I managed to elicit the answer that a spring obeyed such a law of force. I don't think I recognized this while reading Newton at St John's College, by the way; I just thought Newton was having fun finding different force laws for different orbits. (See his Proposition 10.)

When I asked who would like a modern translation of the *Principia,* several people raised hands; so I copied the relevant pages from the Cohen/Whitman version and put them in the library.

6.8. Tuesday, May 4

I decided to review what Zhala had proved for Proposition 4. She seemed happy enough to be relieved of having to say more. In fact she wanted to cut class because her father was visiting.

Given the circle in Figure 6.5, we have

$$F \overset{\text{ult}}{\propto} AC = \frac{AB^2}{AD}, \qquad AB \overset{\text{ult}}{=} \operatorname{arc} AB, \qquad AD = 2R,$$

and therefore

$$F \propto \frac{\text{arc}^2}{R}$$

—and this is an absolute statement, not an "ultimate" one. For Cor. 1, since $V \propto \text{arc}$,

$$F \propto \frac{V^2}{R}.$$

For Cor. 2, since $T \propto R/V$, and $F \propto (V/R)^2$,

$$F \propto \frac{R}{T^2}.$$

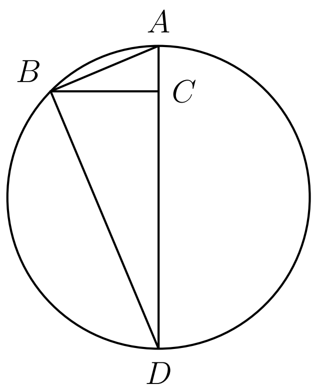

Figure 6.5. Uniform circular motion

The next corollaries are a special case of 7:

$$F \propto \frac{1}{R^{2n-1}} \iff T \propto R^n \iff V \propto \frac{1}{R^{n-1}}.$$

Şule then proved Proposition 6 capably, though I complained that what she wrote on the board did not show the logical connections.

Mehmet Doğan was absent, though he was supposed to present Prop. 7 (on a circular orbit with arbitrary center of force); I did it.

Salih Kanlıdağ presented Proposition 9, on a spiral orbit. He presented all of the steps, but admitted to not knowing what it meant that the figure was "given in shape." I went to the board and distinguished Newton's spiral—our logarithmic spiral—from, say, the spiral of Archimedes. Then Salih got the point.

Fuat had discussed his assignment, Lemma 10, with me, and I had argued that the claim followed from Newton's Lemma 2, since the ellipse was just a circle dilated in one direction. But in class he wasn't ready to make an argument. After class I photocopied for him the relevant pages from the appendix of [17]. Meanwhile, in class, we went ahead with Mehmet Şekercioğlu's presentation of Proposition 10. He was sometimes confused, but classmates gave some help. He was supposed to present also Corollaries 1 and 2, but I guess we skipped those. At the end of class, I made a general comment about how it could be difficult to think at the board, so people sitting should give help or corrections, as they had been, and not just wait for me to do it. I also told Mehmet not to say "Newton says," since we all can read what he says; the point is to tell us the *truth.*

6.9. Thursday, May 6

Working on Proposition 11, elliptic orbits with center of force at a focus, Şule visited my office a couple of times. Fortunately I had notes from working through the proof the night before. There was a mistake in the *Wikipedia* text (now corrected by me); also $Gv \times vP$ is called GvP there. Şule needed to be reminded of the relation of points on an ellipse to the foci; we needed to discuss also how a tangent related to the lines from the focus.

Şule caught me again as I was on the way to the classroom. But *in* class, first Fuat presented Lemma 12. I was dismayed when he went to the board with the photocopy I gave him. At elaborate length he reviewed the definitions of *diameter, conjugate diameter,* and *ordinate* given there. Then he worked through the proof step by step. This means he repeated the tedium of Densmore's presentation, which fails to obtain its equation (4) *immediately* from (2), but just repeats the proof with different letters. But the class paid attention (as well as they ever do, at least). I hadn't actually read Densmore's proof, thinking it looked excessively long. But now I see it's a nice argument.

There were only 15 minutes left, and Şule was keen to present Proposition 11. We stayed a few minutes late so she could finish. She never quite proved $PE = AC$, even though I pointed out the gap; perhaps she was just too excited.

As I was leaving, Ali pointed out that his assignment, Proposition 12, is almost word for word the same as Prop. 11. I suggested he give the alternative argument next time. He admitted to not having read it.

6.10. Tuesday, May 11

Ali proved Prop. 12, using Newton's main proof; he did *not* follow my suggestion of presenting the alternative proof. He did use his own notes, not the text; but he stood directly in front of his writing. When I mentioned this, suggesting that he should explain better what he was doing, he just asked the class in an ironic tone: "Does anybody need this explained?"

Ece followed with Lemma 13, but she had missed the whole point: she had not understood that any point on the parabola could be a vertex. She proved the lemma for the "principal" vertex only. I proved it in general.

Duygu was fine with Lemma 14 and Corollary 1.

Besmir was absent, so he did not present Prop. 13. Since Mihail was supposed to do Cor. 1, I thought he might be able to present the main proposition. He couldn't, so I did it. Mihail couldn't say much about so-called Corollary 1, the converse of Propositions 11–13. Indeed, I didn't know a proof either, except insofar as Prop. 17 is a proof.

Fuat, assigned Cor. 2, was absent. I let Salih Kanlıdağ go ahead with Prop. 14. When he needed Prop. 13, Cor. 2, I got up to observe that the result followed from a part of my proof of Prop. 13 itself that was already on the board. However, Prop. 13 is about parabolas, and Cor. 2 is about all conic sections. I hadn't gone back to check that the same claim followed for the ellipse and hyperbola.

I think Salih skipped Cor. 1 of Prop. 14, though it was part of his assignment.

Mehmet Şekercioğlu said he could do Prop. 15 if I wanted, but there were five minutes left. I asked what he preferred, and he refused to give an opinion. I asked what the class thought, and he observed that they probably wanted to stop for the day; so we did.

After class, this Mehmet asked me: Why aren't all orbits in the universe circular, rather than elliptical? I tried to argue that a circle was a limiting case. I said if we could through a rock fast enough, and there were no air resistance, we could put the rock into orbit: but a *circular* orbit would need just the right speed. But first I drew the wrong picture, the left one of Fig. 6.6; I had forgotten that the center of force was at a *focus.* I corrected to the right figure. Oğuzhan was there; I think he recognized the problem.

But Mehmet Ş. didn't seem to be satisfied. It bothered him that force would *change* as a planet followed its orbit. I tried to suggest that the force still obeyed *one law,* the inverse-square law.

Mehmet said everything happened for a *reason.* I suggested that this was only his assumption. If you assume everything has a reason, then you can find a reason; but it may not be a good one.

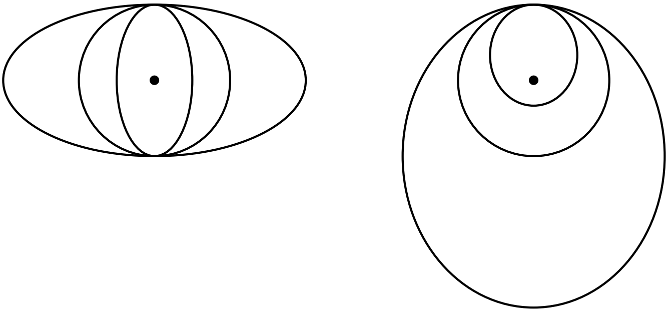

Figure 6.6. Various orbits

6.11. Thursday, May 13

It's the spring festival, and the few students who showed up were happy to cancel class and go back to the festival. I had just been negociating with Ece in my office. She is attending Antalya Algebra Days next week for some reason. Our second exam is on Tuesday, but the conference starts Wednesday, and Ece wanted to travel Monday night with other students. I said she could travel Tuesday night with Ayşe and me, but this was not appealing. She found out that some other METU people were taking a 14:00 bus on Tuesday; could she start the exam earlier than the others on Tuesday, in order to catch that bus?

I said we would finalize our agreement tomorrow; but meanwhile, we would see each other in class. Oh, can I go to the Spring Festival now? Ece asked. I said I didn't give permission for such things; we would just have class. Again, as it happened, we didn't have class, but I did talk a bit about Newton—about how his work showed that the earth and the heavens obeyed the same law. I talked about what might be on the exam: Viète's Law of Homogeneity; Cartesian-style constructions and their equations; proofs as in Newton's first 11 lemmas;

6.12. Tuesday, May 25

The second exam was last Tuesday, and I cancelled Thursday's class to go to Antalya Algebra Days. Today, Duygu in particular asked about the exam: many students were hoping to graduate, but if the exam were graded by catalogue, perhaps these students couldn't graduate. I said they shouldn't worry if they came to class and continued to work. In earlier exams, students had often done my problems better then I expected; this time they did worse. I said I *liked* the last exam, and the students *should* be able to do its problems now; indeed, the analogue of Problem 2 for the ellipse or hyperbola might appear on the final exam, I said.

Mehmet Şekercioğlu finally presented Prop. 15 and its corollary. I asked *where* on the orbit the distance of a planet from the center of force was equal to half the major axis. He didn't know exactly where, but Ali came to the board to show it. For the corollary, Mehmet first draw a concentric circle and ellipse, until he was corrected.

Gökçen presented Prop. 16 and its corollaries. She had still been working with the Motte translation printed out from Wikisource, and she had visited my office, confused. There were a number of mistakes there. I corrected them on line with her; but it was bad of me not to have been reading this translation for mistakes all along. In class, Gökçen was confused about Corollary 2, I don't know why. I drew a diagram for it on the board, as in Figure 6.7.

One proposition was left, 17. I had assigned to Melis, but she was not in class. Well, there was no time left anyway.

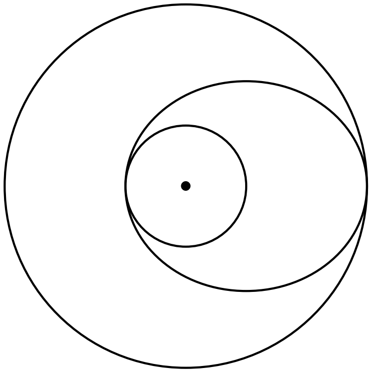

Figure 6.7. Tangent orbits

6.13. Thursday, May 27

Melis didn't come again today. I proved Proposition 17. But the question of how one determines whether the orbit is an ellipse or an hyperbola was not fully resolved. Ali and Oğuzhan in particular were active.

But people asked about the final exam. Salih Acar asked for sample problems: I cited the two exams we have had so far. I admitted I didn't know how to ask problems about §§ 2 and 3 of the *Principia,* except insofar as they concern conic sections. I observed that Newton's ideas about tangents and about finding areas (illustrated in the second exam) continued to be important.

That was the last class.

A. Examinations

These are the examination problems given in the course, along with my solutions and remarks, which I posted on the web after each exam. There were only two exams in Math 303; but at least one student who continued on to Math 304 wanted more exams, so there were three in Math 304.

A.1. Friday, November 6

Problem A.1.1. *What is wrong with the following proof that all triangles are isosceles? [See Figure A.1.]*

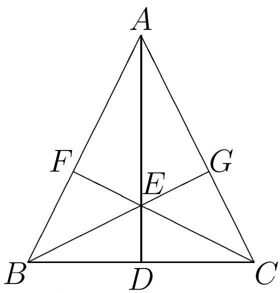

Figure A.1. Are all triangles isosceles?

1. *Let a triangle be given, namely ABC.*
2. *Let BC be bisected at D.*
3. *Let a straight line, DE, be drawn at right angles to BC.*
4. *Let also the straight line AE bisect the angle BAC.*

5. *Let the straight lines BE and CE be drawn.*
6. $BE = CE$.
7. *Let the straight line EF be drawn perpendicular to AB.*
8. *Let the straight line EG be drawn perpendicular to AC.*
9. $AF = AG$ *and* $EF = EG$.
10. $BF = CG$.
11. $AF + FB = AG + GC$.
12. $AF + FB = AB$ *and* $AG + GC = AC$.
13. $AB = AC$*; in particular,* ABC *is isosceles.*

Solution. Step 12 is not justified. In fact, if $AB > AC$, then $AF + FB = AB$, but $AC + GC = AG$.

Remark. 1. The diagram is misleading; but (contrary to what some people seemed to think) the proof never assumes that AED or BEG or CEF is a straight line.

2. Step 4 may *appear* unjustified; however, steps 2, 3, and 4 together say simply that the bisector of angle BAC and the perpendicular bisector of BC meet at E. This style of writing can be seen for example in Euclid's Proposition I.44.

3. The proof does wrongly assume that E lies within the triangle; but the proof can easily be adjusted to the case where E lies outside the triangle. Euclid usually does not bother to consider all possible cases: we noted this for example in Proposition I.7. The real problem is the assumption that either both F and G lie on the triangle, or both lie below the triangle.

Problem A.1.2. *Write English translations of the following words: (a) θεώρημα, (b) πρόβλημα, (c) ἀνάλυσις, (d) συνθέσις, (e) πολύγωνον, (f) τρίγωνον.*

Solution. Theorem, problem, analysis, synthesis, polygon, triangle.

Remark. 1. A *transliteration* of the words into English (or Latin) letters would be *theorêma, problêma, analysis, synthesis, polygônon, trigônon,* but this is not what was asked.

2. The first two words on the list have been discussed in class; these (along with the next two) are also discussed in some notes that I put on the web.

3. The last two words on the list derive from *γωνία angle,* which is apparently related to *γόνυ*; this word shares its meaning, and an Indo-European ancestor, with the English *knee.* (Here is a point where English spelling is useful; if *knee* were spelled phonetically, then its relation with *γόνυ* could not be seen.)

4. As a translation of *τρίγωνον*, I do find the word *trigon* in the Oxford English Dictionary; but the more usual word is of course *triangle.*

Problem A.1.3. *Write the letters of the Greek alphabet in the standard order. Write only the capital letters* or *only the minuscule letters.*

Solution. Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω or *α β γ δ ϵ ζ η θ ι κ λ μ ν ξ ο π ρ ς τ υ ϕ χ ψ ω*.

Problem A.1.4. *Proclus writes:*

> *Every problem and every theorem that is furnished with all its parts should contain the following elements:*
>
> *(1) an* enunciation *(πρότασις),*
> *(2) an* exposition *(or* setting out: *ἔκθεσις),*
> *(3) a* specification *(or* definition of goal: διορισμός*),*
> *(4) a* construction *(κατασκευή),*
> *(5) a* proof *(ἀπόδειξις), and*
> *(6) a* conclusion *(συμπέρασμα).*

Below is the enunciation (in Heath's translation) of Proposition I.6 of Euclid's Elements. *Supply the remaining parts (in your own words, which may or may not be Euclid's).*

> If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Solution. 1. (As above, namely:) If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

2. Let ABC be a triangle in which angles ABC and ACB are equal.

3. We shall show that $AB = AC$.

4. On BA, extended if necessary, let BD be cut off equal to CA.

5. Then triangle DBC is equal to ACB, and therefore D must coincide with A. Consequently, $BA = CA$.

6. Thus we have shown that, if in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Remark. Euclid's proof is a *reductio ad absurdum,* that is, a proof by contradiction. In particular, Euclid first assumes $AB \neq AC$ and *then* finds D. In this case, to which of Proclus's six parts does the hypothesis $AB \neq AC$ belong? I don't know whether Proclus considers this question.

Problem A.1.5. Without *using Euclid's method of "application," prove Proposition I.8 of the* Elements, *whose enunciation is,*

> If two triangles have two sides equal to two sides respectively, and have also the base equal to the base, they will also have

the angles equal whch are contained by the equal straight lines.

Solution. Suppose ABC and DEF are triangles such that $AB = DE$, $BC = EF$, and $AC = DF$. We shall show that angles ABC and DEF are equal. To this end, let AG be dropped perpendicular to BC, extended if necessary [by I.12]. On EF, extended if necessary, cut off EH equal to BG [by I.3]. Erect HK perpendicular to EF [by I.11] and equal to AG [by I.3 again]. Then $EK = AB$ and angles KEF and ABC are equal [by I.4], and similarly, since $HF = GC$, we have $FK = CA$. Hence $EK = ED$ and $FK = FD$. Therefore K and D coincide [by I.7], and in particular, angles DEF and ABC are equal.

Now, we have used two propositions [namely I.11 and 12] that Euclid proves by means of I.8. However, alternative proofs are as follows.

If A does not lie on the straight line BC, then by drawing a circle with center A that cuts the line, we may assume B and C have been chosen so that $AB = AC$. Draw an equilateral triangle BCD (on the opposite side of BC from A) [by I.1]. Draw the straight line AD, which cuts BC at a point E. Then angles BAD and CAD are equal [by I.5 and 4], and therefore angles AEB and AEC are equal [again by I.4], so the latter angles are right. Therefore AE has been dropped perpendicular to AB.

If A does lie on BC, we may still assume $AB = AC$. Draw an equilateral triangle BCD and straight line AD. Then angles BAD and CAD are equal [by I.5 and 4], so they are right. Thus AD has been erected perpendicular to BC.

Remark. 1. It is not necessary to name the propositions used.

2. Some people argued by contradiction that if (in the notation above) angle ABC is greater than DEF, then BC must be greater than EF. This is Proposition I.24; but I.24 relies on I.23, which in turn relies on I.8. It is not clear to me that there is a way to prove I.24 without first proving I.8.

3. One person suggested an interesting argument that I understand as follows. If angle ABC is greater than DEF, then inside the former angle, there must be an angle ABG equal to DEF. We may then assume $BG = BC = EF$. But then $GA = FA$ [by I.4], so we have violated I.7, which is absurd; therefore $ABC = DEF$. Now, if this argument is valid, then what is the point of I.3? If straight line AB is greater than straight line C, why does Euclid not declare that there must be a part of AB, namely AE, that is equal to C? Why does Euclid feel the need to *construct* AE?

Problem A.1.6. *In triangle ABC, suppose BC is bisected at D, and straight line AD is drawn. Assuming AB is greater than AC, prove that angle BAD is less than DAC.*

Solution. Extend AD to E so that $DE = DA$. Then angles DEC and DAB are equal, and $CE = BA$ [by I.4]. But then angle CAE is greater than CEA [by I.18], so $CAD > DAB$.

Remark. I think the argument just given is the best of several variants that were found by different people. The argument I had thought originally of was more complicated: Since angle BDA must be greater than ADC, inside angle BDA we can construct angle ADE equal to ADC, with $DE = DC$. Then BE is parallel to AD [why?], so E lies outside triangle ABD. Therefore angle BAD is less than EAD; but the latter is equal to DAC.

A.2. Make-up exam

This was given Friday, January 22, to Rashad and Tolga.

Problem A.2.1. *What is wrong with the following proof that angles have no size?—:*

1. *Let an angle K be given [Figure A.2].*

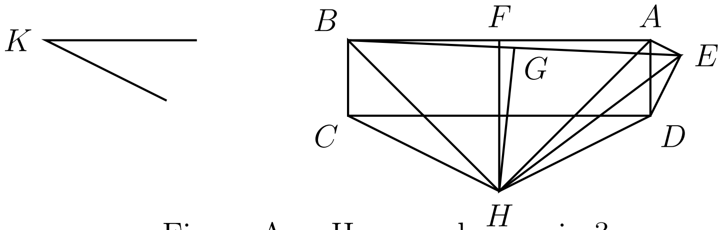

Figure A.2. Have angles no size?

2. *Let a rectangle be given, namely $ABCD$.*
3. *Let angle EDA be equal to K.*
4. *Let DE be made equal to DA.*
5. *Suppose the perpendicular bisectors FH of AB and GH of BE meet at H.*
6. *Let the straight lines HC, HB, HF, HG, HA, HE, and HD be drawn.*
7. *$HB = HA$.*
8. *$HB = HE$.*
9. *$HA = HE$.*
10. *Triangles HAD and HED are equal in all respects.*
11. *In particular, angle HDA is equal to HDE.**
12. *Angle EDA has no size.*
13. *Therefore K has no size.*

*The exam had HDA for HDE here. This went unnoticed till April 29, 2011.

Problem A.2.2. *Write English translations of the following words:*

(a) γραμμή, (b) κύκλος, (c) κέντρον, (d) τρίγωνον,
(e) περιφέρεια, (f) γεωμετρία.

Problem A.2.3. *What are five postulates of Euclid's* Elements*?*

Problem A.2.4. *Proclus writes:*

> *Every problem and every theorem that is furnished with all its parts should contain the following elements:*
>
> *(1) an* enunciation *(πρότασις),*
> *(2) an* exposition *(or* setting out: *ἔκθεσις),*
> *(3) a* specification *(or* definition of goal: *διορισμός),*
> *(4) a* construction *(κατασκευή),*
> *(5) a* proof *(ἀπόδειξις), and*
> *(6) a* conclusion *(συμπέρασμα).*

Below is the enunciation (in Heath's translation) of Proposition I.7 of Euclid's Elements. *Supply the remaining parts (in your own words, which may or may not be Euclid's).*

> Given two straight lines constructed on a straight line [from its extremities] and meeting in a point, there cannot be constructed on the same straight line [from its extremities], and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

Problem A.2.5. *From I.7, by the method of "application," Euclid proves I.8, whose enunciation is:*

> If two triangles have two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straighte lines.

Assuming this proposition, but without using the method of "application," prove the following (which is part of the enunciation of I.4):

> If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base.

Problem A.2.6. *In a triangle ABC, suppose D is chosen on side AB, and E is chosen on AC, so that DE is parallel to BC. Suppose straight line DF is drawn parallel to AC, and CF is drawn parallel to AB, and DF and CF meet at F. Similarly, suppose BG is drawn parallel to AC, and EG is drawn parallel to AB, and BG and EG meet at G. Let straight line GF be drawn.*

Prove that GF is parallel to BC. You may use only propositions from Book I of the Elements.

A.3. Tuesday, January 12

Problem A.3.1. *In the 8th century* B.C.E.*, the colony of Cumae (*Κύμη*) was founded, near what is now Naples, by settlers from Euboea (Eğriboz), and also from Cyme (*Κύμη*) in western Anatolia near what is now Aliağa [30, 54]. From the Greek alphabet as used in Cumae, the Latin alphabet was ultimately derived; this came to have 23 letters:*

A B C D E F G H I K L M N O P Q R S T V X Y Z.

In the year 863 C.E.*, a monk from Salonica named Cyril invented the so-called Glagolitic alphabet in order to translate holy scripture from Greek into Old Bulgarian. Soon after that, the simpler Cyrillic alphabet was invented [28].* After some*

*Many alphabets can be seen in [26].

changes (such as the abolition of a few letters by the Soviet government in 1918), the Cyrillic alphabet became the 33-letter Russian alphabet of today:

А Б В Г Д Е Ё Ж З И Й К Л М Н О П
Р С Т У Ф Х Ц Ч Ш Щ Ъ Ы Ь Э Ю Я.

This alphabet retains 19 of the 24 letters of the Greek alphabet, in their original order, though not always in the original form. What are *the 24 letters of the Greek alphabet?*

Solution. Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω, or $\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ o\ \pi\ \rho\ \varsigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega$.

Remark. Most people seem to have learned the alphabet for this exam. If this had been so on the first exam, I may not have asked for the alphabet on *this* exam.*

Problem A.3.2. *Does a square have a ratio to its side? Explain.*

Solution. No, since no multiple of the side can exceed the square.

Remark. This problem alludes to Definition 4 of Book V of the *Elements:*

> Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another.

Euclid does not seem to *refer* to this definition later; but (as we discussed in class) he *uses* the definition implicitly, in Proposition V.16 for example, where there is an unstated assumption

*Note added, September 18, 2013: A horrible possibility that I did not consider is that some students were able to cheat.

that A and C have a ratio, and (therefore) B and D have a ratio. In his "quadrature of the parabola," discussed on the last day of class, Archimedes assumes that, if two areas are unequal, then their difference *has a ratio* (in the sense of Euclid) to either of the areas.

Problem A.3.3. *Suppose a magnitude A has a ratio to a magnitude B, and a magnitude C has a ratio to a magnitude D. What does it mean to say that A has the* same *ratio to B that C has to D (according to Definition 5 of Book V of Euclid's* Elements*)?*

Solution. If equimultiples mA and mC of A and C be taken, and other equimultiples nB and nD of B and D be taken, then

$$mA > nB \text{ if and only if } mC > nD,$$
$$mA = nB \text{ if and only if } mC = nD,$$
$$mA < nB \text{ if and only if } mC < nD.$$

Remark. The definition of ratio is perhaps the most important sentence in Euclid. Euclid of course does not use special notation for a multiple of a magnitude.

Problem A.3.4. *Suppose a straight line AB is bisected at C, and another point, D, is chosen on AB. What is the relation between the squares on AC and CD and the rectangle contained by AD and DB?*

Solution. $AC^2 = CD^2 + AD.DB$ [by Euclid's II.5].

Problem A.3.5. *In the diagram [Figure A.3], BAC is the diameter of a circle, A is the center, and AD is at right angles to BC. Straight line DC is drawn. From a point E on the*

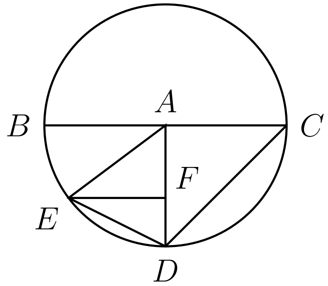

Figure A.3. The swing of a pendulum

*circumference between B and D, the straight line EF is drawn at right angles to AD, and EA and ED are drawn. Show that the square on DE has the same ratio to the square on DC that the straight line DF has to DA. (*Suggestion: *express DE^2 and DC^2 in terms of DF, FA, and DA.)*

Solution. Just compute: $DC^2 = 2DA^2$, while

$$\begin{aligned} DE^2 = DF^2 + FE^2 &= DF^2 + EA^2 - FA^2 \\ &= DF^2 + DA^2 - FA^2 \\ &= 2DF^2 + 2DF.FA = 2DF.DA, \end{aligned}$$

so $DE^2 : DC^2 :: 2DF.DA : 2DA.DA :: DF : DA$.

Remark. The equation $DA^2+DF^2 = 2DF.DA+FA^2$ happens to be the symbolic expression of Euclid's Proposition II.7. I obtained this problem from Isaac Newton, who writes in the *Principia,* in the scholium after the Laws of Motion:

> It is a proposition very well known to geometers that the velocity of a pendulum at the lowest point is as the chord of the arc which it describes in falling.

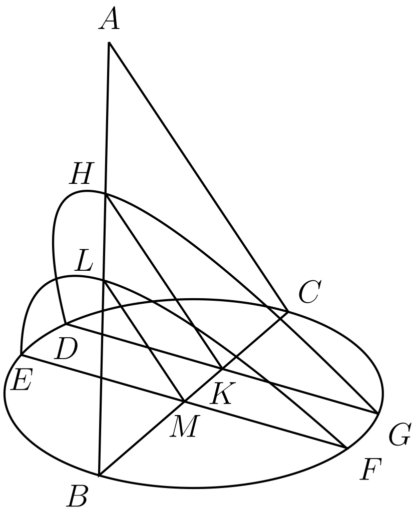

Figure A.4. Two parabolas in a cone

Problem A.3.6. *In the diagram [Figure A.4], ABC is an axial triangle of a cone whose base is the circle $CDEBFG$, and DKG and EMF are at right angles to BC. Planes through DKG and EMF cut the cone, making sections DHG and ELF, with diameters HK and LM, respectively; and these diameters are parallel to AC. The* ***parameters*** *(the "upright sides" or* latera recta*) of the sections are not shown; but let them be HN and LP. What is the ratio of HN to LP (in terms of straight lines that* are *shown in the diagram)?*

Solution. Since $HN : HA :: BC^2 : BA.AC$ and $LP : LA :: BC^2 : BA.AC$ [by I.11 of Apollonius], we have $HN : HA :: LP : LA$, and alternately

$$HN : LP :: HA : LA.$$

Remark. One may alternatively observe that $DK^2 = HN.HK$, but also $DK^2 = BK.KC$, and similarly for EM. Hence

$$DK^2 : EM^2 :: HN : LP \,\&\, HK : LM, \qquad (*)$$

but also

$$\begin{aligned} DK^2 : EM^2 &:: BK : BM \,\&\, KC : MC \\ &:: HK : LM \,\&\, HA : LA, \end{aligned}$$

and therefore $HN : LP :: HA : LA$. Now, from $(*)$, one might write

$$\begin{aligned} HN : LP &:: DK^2 : EM^2 \,\&\, LM : HK \\ &:: DK^2.LM : EM^2.HK; \end{aligned}$$

but this isn't the best answer. A better answer is $HN : LP :: CK : CM$, but this still refers to the particular choice of base for the cone, when the parabolas themselves do not depend on this choice.

Problem A.3.7. *We know that an ellipse or an hyperbola has two "conjugate" diameters, each diameter being situated ordinatewise with respect to the other. A parabola cannot have conjugate diameters in this sense. Nonetheless, suppose, in the diagram [Figure A.5], AB is the diameter of a parabola, and AC is drawn ordinatewise, and AC is also the diameter of another parabola, and AB is situated ordinatewise with respect to AC. Suppose the two parabolas meet at D (as well as at A). Let the respective ordinates DB and DC be dropped. Finally, suppose the parabola with diameter AB has parameter E (not shown), and the parabola with diameter AC has parameter F.*

Show that

$$E : AC :: AC : AB, \qquad AC : AB :: AB : F.$$

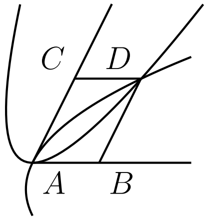

Figure A.5. Two intersecting parabolas

*(*Remark. *It follows then that E is to F as the* cube *on AC is to the cube on AB. In particular, if E is twice F, then the cube on AC is double the cube on AB. According to Eutocius in his* Commentary on Archimedes's Sphere and Cylinder, *Menaechmus discovered this method of "duplicating" the cube, along with another method involving a parabola and a hyperbola. This work is the earliest known use of conic sections. For Menaechmus however, the angle BAC would have been right.)*

Solution. Since $AB.E = BD^2 = AC^2$, we have $E : AC :: AC : AB$; the other proportion is similar.

Problem A.3.8. *In the triangle ABC shown [Figure A.6], FG is parallel to DC, and DE is parallel to AG. Show that AC is parallel to FE. (You may use the theory of proportion developed in Books V and VI of the* Elements. *In that case, you will probably want to use* alternation: *if $A : B :: C : D$, then $A : C :: B : D$. You may use also that if $A : B :: E : F$ and $B : C :: D : E$, then $A : C :: D : F$. Alternatively, it is possible to avoid the theory of proportion by showing, as a lemma, that, in the diagram, FE is parallel to AC if and only if the parallelogram bounded by BF and BC, in the angle B, is equal to the parallelogram bounded by BE and BA. Or maybe you can find another method. In modern terms, this problem*

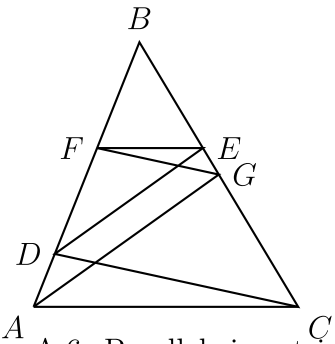

Figure A.6. Parallels in a triangle

can be set in a two-dimensional vector-space; but if the scalar field of that space is non-commutative, then the claim is false.)

Solution. Because of the parallels, we have

$$BF : BD :: BG : BC, \qquad BD : BA :: BE : BG;$$

therefore [by the suggested result, which is V.23 of Euclid] $BF : BA :: BE : BC$, which yields the parallelism of FE and AC.

Remark. I learned this short proof from some students' papers. I had previously found a longer argument, which *did* use alternation.

Really, Euclid's VI.2 gives us only (for example) $DF : FB :: CG : GB$; this is equivalent to $DB : FB :: CB : GB$ by V.17 and 18.

As noted, we don't really need to use proportions, just that, in the diagram here [Figure A.7], the parallelograms $ABEG$ and $BCKF$ are equal (by cutting and pasting) if and only if FE is parallel to AC. Let's use $BA.BE$ and $BF.BC$ to denote these parallelograms respectively. In the problem then,

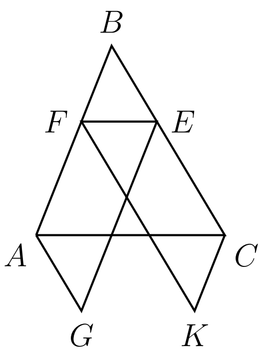

Figure A.7. Parallelograms

we have $BF.BC = BG.BD = BA.BE$, so $AE \parallel BE$. This problem is inspired by Descartes, who, in his *Geometry,* observes that, if (in the original diagram) BF is a unit length, and $BG = a$, while $BD = b$, then we can define the product ba as (the length of) BC. Descartes does not show that the multiplication so defined is commutative. But it *is* commutative, by this problem. Indeed, if $BE = BF$, then $BA = ab$, but also $BA = BC$, so $ab = ba$.

However, if you know about the skew-field $\mathbb{H}$ of *quaternions,* then suppose the diagram sits in the vector-space $\mathbb{H}^2$ as shown below [Figure A.8]. Then the assumptions of parallelism in the problem hold here, since for example $(0, \mathrm{ij}) - (\mathrm{i}, 0)$ is a scalar multiple of $(0, \mathrm{j}) - (1, 0)$. However, $(0, \mathrm{ij}) - (\mathrm{ji}, 0) = \mathrm{ij}(-1, 1)$, which is not a scalar multiple of $(0, 1) - (1, 0)$.

Bonus. *How can this exam and this course be improved? (Responses may be submitted also by email in the next few days:* `dpierce@metu.edu.tr`*. Meanwhile,* iyi çalışmalar; ondan sonra, iyi tatiller!*)*

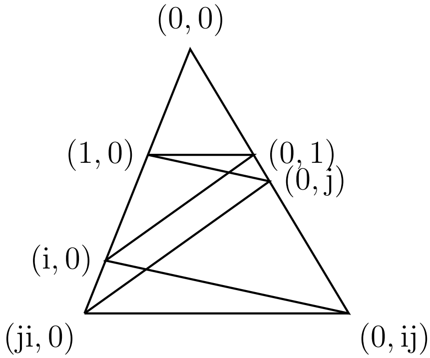

Figure A.8. Parallels in a triangle again

A.4. Tuesday, March 30

You may use modern notation in your work; but Problems A.4.2 and A.4.3 should involve diagrams.

Problem A.4.1. *A straight line is cut into equal and unequal segments. What is the relationship between the square on the half and the rectangle contained by the unequal segments?*

Solution. The square exceeds the rectangle by the square on the straight line between the points of section.

Remark. This problem is based on Proposition II.5 of Euclid's *Elements.* The language follows the style of Heath's translation of Euclid (on the course webpage).

Problem A.4.2. *A square is equal to three roots and twenty-eight dirhams. What is the root? Give a geometrical justification of your answer (as Muḥammad ibn Mūsā al-Khwārizmī or Thābit ibn Qurra did).*

Solution. In Figure A.9, the root is AB; $AC = 3$; and D bisects AC. Then

$$DB^2 = 28 + DC^2 = 28 + \left(\frac{3}{2}\right)^2 = \frac{121}{4},$$
$$DB = \frac{11}{2},$$
$$AB = AD + DB = \frac{3}{2} + \frac{11}{2} = 7;$$

so the root is 7.

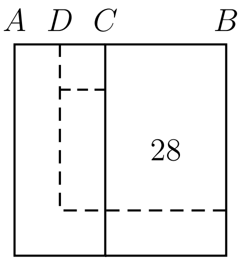

Figure A.9. Analysis of a square

Remark. Euclid's Proposition II.6 is behind this problem.

Problem A.4.3. *Suppose a cube and nine sides are equal to ten. Find the side by taking the intersection of two conic sections (as Omar Khayyām did). It is preferable if one of those sections is a circle.*

Solution. [*Analysis:*]

$$x^3 + 9x = 10,$$
$$x^3 = 10 - 9x,$$
$$\frac{x^2}{9} = \frac{10/9 - x}{x}, \qquad (\dagger)$$
$$\frac{x}{3} = \frac{y}{x} = \frac{10/9 - x}{y}, \qquad (\ddagger)$$

$$x^2 = 3y \quad \& \quad y^2 = x\left(\frac{10}{9} - x\right).$$

[*Synthesis:*] As in Figure A.10, let ABC be a semicircle with

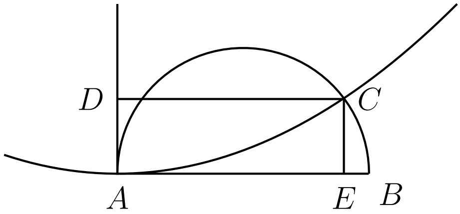

Figure A.10. Circle and parabola

diameter 10/9, and let AD, perpendicular to AB, be the axis of a parabola with parameter 3. The semicircle and parabola intersect at a point C (as well as at A). Let CE be dropped perpendicular to AB; and CD, to AD. Then $AE = CD$; either of these is the desired "side." Indeed,

$$\begin{gathered} CD^2 = 3AD, \\ CD : 3 :: AD : CD :: EC : AE :: EB : EC, \\ AE^2 : 9 :: CD^2 : 9 :: EB : AE :: \left(\frac{10}{9} - AE\right) : AE, \\ AE^3 = 10 - 9AE, \\ AE^3 + 9AE = 10. \end{gathered}$$

Remark. (i). In the solution, *analysis* and *synthesis* are used in the sense attributed to Theon (presumably Theon of Smyrna, that is, İzmir) by Viète at the beginning of Chapter 1 of the *Introduction to the Analytic Art.* In his solutions of cubic equations, Omar Khayyām gives only the synthesis; we can only speculate whether he had some sort of analysis like ours.

(ii). In our analysis, equations (†) and (‡) could have been

$$x^2 = \frac{10-9x}{x},$$
$$x = \frac{y}{x} = \frac{10-9x}{y},$$

yielding the parabola given by $y = x^2$ and the ellipse given by $y^2 = x(10-9x)$. This is why the problem says, "It is preferable if one of those sections is a circle."

(iii). I think it is better to understand the circle through the equation $y^2 = x(10/9 - x)$ than to convert this equation to the more usual modern form,

$$y^2 + \left(x - \frac{5}{9}\right)^2 = \left(\frac{5}{9}\right)^2.$$

Problem A.4.4. *Again, a cube and nine sides are equal to ten.*

1. *Find the side numerically, as the difference of the cube roots of a* binomium *and an* apotome, *by Cardano's method (really Tartaglia's method); your steps should be clearly justifiable.*
2. *The side is in fact a whole number; which one?*

Solution. 1. We have to solve $x^3 + 9x = 10$. We let $x = u - v$, so

$$x^3 = u^3 - v^3 - 3uv(u-v) = u^3 - v^3 - 3uvx.$$

So we let

$$u^3 - v^3 = 10, \qquad uv = 3,$$

which we can solve:

$$u^6 - u^3v^3 = 10u^3,$$
$$u^6 - 27 = 10u^3,$$
$$u^3 = \sqrt{5^2 + 27} + 5 = 2\sqrt{13} + 5,$$
$$v^3 = \frac{3^3}{2\sqrt{13} + 5} = 2\sqrt{13} - 5.$$

Therefore

$$x = \sqrt[3]{2\sqrt{13} + 5} - \sqrt[3]{2\sqrt{13} - 5}.$$

2. $x = 1$.

Remark. (i). Cardano does give a formula for finding x, without clear explanation. However, this problem said "steps should be clearly justifiable"; so for full credit, the answer should be *derived,* as above, not just obtained from a memorized formula. Some people who tried to memorize, remembered wrongly.

(ii). Of course, the solution above did rely on the (memorized) quadratic formula. Memory does have its uses.

(iii). Note here that u^3 could have been $-2\sqrt{13}+5$; but x in the end would have been the same. Two other values of x can be obtained by considering *complex* cube roots; but Cardano does not know about these.

Problem A.4.5. *A square-square, twelve squares, and thirty-six are equal to seventy-two sides. In finding the side by Cardano's method (really Ferrari's method), you first solve a cubic equation.*

1. *Obtain that cubic equation.*

2. *Convert that cubic equation to an equation of the form "cube equal to roots and number."*
3. *The cubic equation in (1) should have 6 as a root. Use this to find the side in the original fourth-degree equation.*

Solution. 1.

$$x^4 + 12x^2 + 36 = 72x,$$
$$(x^2 + 6)^2 = 72x,$$
$$(x^2 + 6 + t)^2 = 2tx^2 + 72x + t^2 + 12t,$$
$$2t(t^2 + 12t) = 36^2 = 2^4 3^4,$$
$$t^3 + 12t^2 = 2^3 3^4 = 648.$$

2. Let $t = s - 4$; then

$$s^3 - 48s + 12 \cdot 16 - 64 = 2^3 3^4,$$
$$s^3 - 48s = 2^3 3^4 + 2^6 - 2^6 3 = 2^3(3^4 - 2^4) = 8 \cdot 65 = 520.$$

3.

$$\begin{aligned}(x^2 + 12)^2 &= 12x^2 + 72x + 108,\\ &= 12(x^2 + 6x + 9)\\ &= 12(x + 3)^2,\end{aligned}$$
$$x^2 + 12 = 2\sqrt{3}(x + 3),$$
$$x^2 = 2\sqrt{3} - 6(2 - \sqrt{3}),$$
$$x = \sqrt{3} + \sqrt{3 - 6(2 - \sqrt{3})} = \sqrt{3} + \sqrt{6\sqrt{3} - 9}.$$

Remark. If we believe in negative numbers, then from $(x^2 + 12)^2 = 12(x + 3)^2$ we should obtain $x^2 + 12 = \pm 2\sqrt{3}(x + 3)$; but the negative sign here leads to a negative value of x. The problem asks for the "side," which is implicitly positive.

A.5. Tuesday, May 18

Problem A.5.1. *The ellipse AEB [Figure A.11] is determined as follows. Triangle ABC is given, the angle at A being*

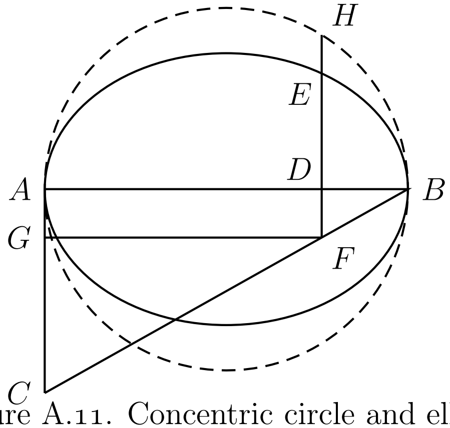

Figure A.11. Concentric circle and ellipse

right. If a point D is chosen at random on AB, and DE is erected at right angles to AB, then E lies on the ellipse if (and only if) the square on DE is equal to the rectangle $ADFG$ (which is formed by letting ED, extended as necessary, meet BC at F). Let also the circle AHB with diameter AB be given.

Find h (in terms of the given straight lines) such that h is to AB as the ellipse is to the circle. Prove that your answer is correct, using Newton's lemmas as needed.

Remark. The ellipse appears to result from contracting the circle in one direction. If this is so, then by Newton's Lemma 4, the ratio of ellipse to circle is the factor of contraction, which should be DE/DH. So one should find this ratio and check that it is indeed independent of the choice of D.

Two students solved this problem perfectly. Five others used without proof a rule for the area of an ellipse; but we do not officially have such a rule, and in fact the point of this problem is to establish this rule.

Solution. By construction and the similarity of the triangles BDF and BAC,

$$DE^2 = ADFG = AD \times DF = AD \times DB \times \frac{AC}{AB}.$$

In the circle,

$$DH^2 = AD \times DB.$$

Let h be a mean proportional of AB and AC, so

$$h^2 = AB \times AC, \qquad \frac{AC}{AB} = \frac{h^2}{AB^2}.$$

Then

$$\frac{DE^2}{DH^2} = \frac{AC}{AB}, \qquad \frac{DE}{DH} = \frac{h}{AB}.$$

If we inscribe series of parallelograms in the ellipse and circle, all of the same breadth, then corresponding parallelograms will be to each other as DE to DH, that is, h to AB. Therefore this is the ratio of the ellipse to the circle [by Newton's Lemma 4].

Problem A.5.2. *We have used without proof Propositions I.33 and 49 of the* Conics *of Apollonius. This problem is an opportunity to prove those propositions, using the techniques of Descartes and Newton as appropriate.*

A straight line ℓ (not shown), a curved line ABE [Figure A.12], and a straight line AC are given such that, whenever a point B is chosen at random on ABE, and straight line BC is dropped perpendicular to AC, then the square on BC is equal to the rectangle bounded by ℓ and AC. So ABE is a parabola with axis AC.

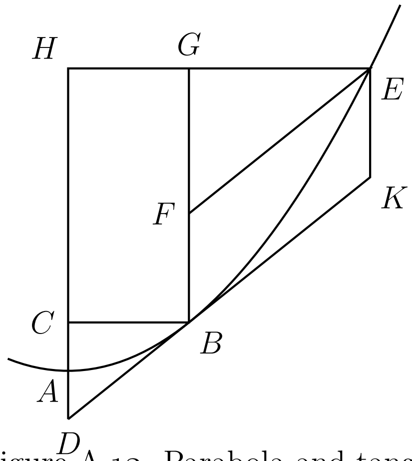

Figure A.12. Parabola and tangent

Let B now be fixed; so we may write $BC = a$ and $AC = b$. Extend CA to D so that $AD = AC$. Draw straight line DBK, and let $c = BD$.

Let a point E be chosen at random on the parabola ABE. Draw straight lines BF parallel to AC, and EF parallel to BD.

1. *Show that the parabola ABE must indeed lie all on one side of DBK.*
2. *Show that the square on EF varies as BF, and find m (in terms of a, b, and c only) such that $m \times BF$ is equal to the square on EF. For your computations, let $x = EF$ and $y = BF$.*
3. *Explain why BD is tangent to the parabola at B.*

Remark. One approach to (a) is showing that E lies above K. The height of E above D is the length of DH; by similarity of triangles, the height of K above D is $2b/a$ times EH. The point of using DH and EH is that we know how their lengths are related. Two students solved this problem perfectly; one other was partially successful.

In (b), we want to find x^2/y in terms of fixed magnitudes. We have one equation, $EH^2 = \ell \times AH$, and we can write this in terms of x and y (and fixed magnitudes) by using the similar triangles BCD and EGF. Three students solved this problem completely; two others got halfway there.

For (c), one student showed that DB is the only straight line passing through B and meeting AD that meets the parabola exactly once. A number of students observed that DB does meet the parabola just once; but this is not enough to establish that DB is a tangent. Note also that BG also meets the parabola exactly once, but is not a tangent.

Solution.

1. Assuming KE is parallel to AC, drop a perpendicular KL to AC. We want to show $DH \geqslant DL$ or $AH \geqslant AL$. We have

$$AH = \frac{EH^2}{\ell}, \qquad DL = LK \times \frac{2b}{a} = EH \times \frac{2b}{a},$$

so

$$\begin{aligned} \ell \times (DH - DL) &= EH^2 + b\ell - EH \times \frac{2b\ell}{a} \\ &= EH^2 + a^2 - EH \times 2a \\ &= (EH - a)^2, \end{aligned}$$

which is positive when E is not B; so $DH > DL$.

2. We have $EH^2 = \ell \times AH$. Since $EG = ax/c$ and $GF = 2bx/c$, this means

$$\Big(a + \frac{ax}{c}\Big)^2 = \ell\Big(y + \frac{2bx}{c} + b\Big),$$
$$a^2 + \frac{2a^2x}{c} + \frac{a^2x^2}{c^2} = \ell y + \frac{2b\ell x}{c} + b\ell,$$

and since $a^2 = \ell b$, we have

$$\frac{a^2x^2}{c^2} = \ell y, \qquad x^2 = \frac{c^2}{a^2}\ell y, \qquad m = \frac{c^2}{b}.$$

3. In the figure, as E approaches B, EK varies as BK^2. Therefore EK/BK varies as BK, so the angle EBK ultimately vanishes.

A.6. Saturday, June 12

Problem A.6.1. *This problem is about the cubic equations*

$$x^3 + 3x^2 = 6x + 17, \tag{§}$$
$$t^3 = 9t + 9. \tag{¶}$$

A. *Explain the relation between the solutions of* (§) *and* (¶).

B. *For one of* (§) *and* (¶), *find a solution geometrically, by intersecting conic sections (as Omar Khayyam does).*

C. *Find* three *solutions in the same way (some might be negative).*

D. *Find a solution of* (§) *or* (¶) *numerically (in the manner suggested by Cardano); your steps should be justifiable. Your answer will involve square roots of negative numbers.*

Solution. **A.** The substitution $x = t - 1$ converts (§) into (¶); so x is a solution to (§) if and only if $x + 1$ is a solution to (¶)).

B. From (¶) we have

$$\frac{t^2}{9} = \frac{t+1}{t}, \qquad \frac{t}{3} = \frac{y}{t} = \frac{t+1}{y}$$

for some y; that is, we can solve (¶) by simultaneously solving $t/3 = y/t$ and $y/t = (t+1)/y$, that is,

$$t^2 = 3y, \qquad\qquad y^2 = t(t+1).$$

These equations define a parabola and a hyperbola, respectively, as below [Figure A.13]. Then AB is a solution to (¶).

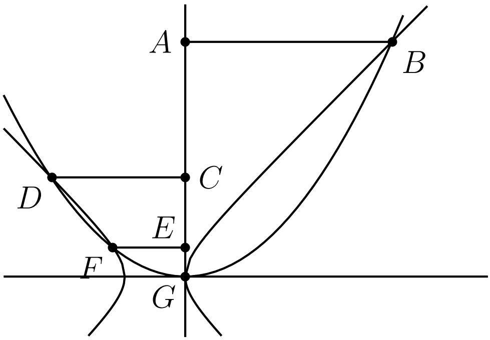

Figure A.13. Intersecting parabola and hyperbola

C. The negative solutions of (¶) are CD and EF. (The parabola and hyperbola intersect also at G, but no solution to (¶) corresponds to this, since the corresponding value of y is 0.)

D. Let $t = u + v$; then

$$t^3 = 3uvt + u^3 + v^3.$$

Then (¶) holds, provided $uv = 3$ and $u^3 + v^3 = 9$. Solving these, we have

$$u^6 + u^3v^3 = 9u^3,$$
$$u^6 + 27 = 9u^3,$$

$$u^3 = \frac{9}{2} \pm \sqrt{\frac{81}{4} - 27} = \frac{9 \pm 3\sqrt{-3}}{2}.$$

So if u is a cube root of $(9+3\sqrt{-3})/2$, then one solution to (¶) is $u + 3/u$.

Remark. Cardano could not give a meaning to the solution we found in the last part; today we can, and the three choices of the cube root give the three solutions found geometrically earlier.

Problem A.6.2. *This problem shows that every line through the center of an ellipse is a diameter with certain properties. The method is based on Apollonius; but the algebraic geometry of Descartes makes some simplifications possible. Straight line*

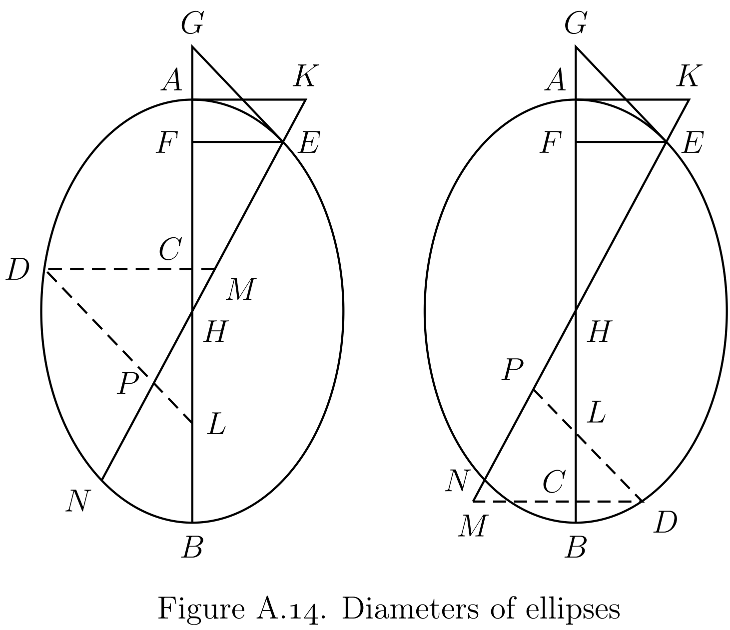

Figure A.14. Diameters of ellipses

AB is given, and angle BAK is given. The point C moves *along AB, and as it moves, straight line CD remains parallel to AK. But D moves along DC as C moves, so that D traces out a curvilinear figure ADB, as shown [Figure A.14] with two possible positions of DC.*

*Recall that the curvilinear figure ADB is an **ellipse** with **diameter** AB and **ordinates** parallel to AK if and only if*

$$CD^2 \propto AC \times CB \qquad (\|)$$

(that is, the square on CD varies as the rectangle formed by AC and CB).

Let E be chosen at random on ADB, and let straight line EF be drawn parallel to KA, meeting AB at F. Let straight line EG be drawn, meeting BA extended at G so that

$$\frac{AG}{GB} = \frac{AF}{FB}. \qquad (**)$$

Let H be the midpoint of AB, and let straight line HE be drawn and extended to meet AK at K. Let L be taken on AB (extended if necessary) so that straight line DL is parallel to GE. Finally, let M be the point of intersection of DC and HK (both extended if necessary).

For computations, let

$$AH = b, \quad EF = c, \quad HF = d, \quad CD = x, \quad CH = y.$$

Also, let a be such that

$$\frac{a^2}{b^2} = \frac{EF^2}{AF \times FB} = \frac{c^2}{b^2 - d^2}. \qquad (\dagger\dagger)$$

***A.** Show that $(\|)$ holds if and only if*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (\ddagger\ddagger)$$

B. *Find HG in terms of b and d.*
C. *Show that (‖) holds if and only if*

$$\triangle CDL = \triangle AHK - \triangle CHM. \tag{§§}$$

(Angle BAK is not assumed to be a right angle; but the computations can be performed as if it were.)

D. *Assuming (‖) holds (and hence (§§) holds, for* all *possibilities for C), show*

$$\triangle AHK = \triangle GHE.$$

E. *Assume (‖) holds. Let EH be extended to meet the ellipse again at N, and let EN meet DL (extended as necessary) at P. Show that the curvilinear figure ADB is an ellipse with diameter EN whose ordinates are parallel to EG. (You will probably want to use part **C**, translated appropriately.)*

Solution. **A.** If (‡) holds, then in particular it holds when C is F. Therefore (‡) is equivalent to

$$\frac{CD^2}{AC \times CB} = \frac{EF^2}{AF \times FB} = \frac{a^2}{b^2},$$
$$\frac{x^2}{b^2 - y^2} = \frac{a^2}{b^2},$$
$$b^2x^2 = a^2b^2 - a^2y^2,$$

which is equivalent to (‖).

B. Let $HG = e$. Then (§) becomes

$$\frac{e-b}{e+b} = \frac{b-d}{b+d},$$

which yields $e = b^2/d$.

C. Since $CDL \sim FEG$, and

$$FEG = \frac{1}{2}\Big(\frac{b^2}{d} - d\Big)c,$$

we have

$$CDL = \frac{x^2}{c^2}FEG = \frac{x^2}{2c}\Big(\frac{b^2}{d} - d\Big).$$

We assume angle BAK is right; otherwise, we can just multiply throughout by its sine.) Also AHK and CHM are both similar to FHE, which is $cd/2$; so

$$AKH - CHM = \frac{cd}{2}\Big(\frac{b^2}{d^2} - \frac{y^2}{d^2}\Big).$$

So $(**)$ holds if and only if

$$\begin{gathered}\frac{x^2}{c}\Big(\frac{b^2}{d} - d\Big) = c\frac{b^2 - y^2}{d},\\ x^2(b^2 - d^2) = c^2(b^2 - y^2),\\ b^2x^2 = a^2(b^2 - y^2),\end{gathered}$$

which is equivalent to $(\|)$.

D. In $(**)$, let C be F; then the equation becomes

$$FEG = AHK - FHE,$$

so $AHK = FEG + FHE = GHE$.

E. By part **C**, it is enough to show

$$PDM = EHG - PHL.$$

We have

$$\begin{aligned}PDM &= CDL + CHM - PHL\\ &= AHK - PHL && [\text{by } (**)]\\ &= EHG - PHL && [\text{by } \mathbf{D}].\end{aligned}$$

Bonus. *What are your suggestions for improving the course?*

Geldiğiniz için teşekkürler. İyi tatiller!

B. Student comments

Here are the comments invited by the "bonus" questions on the final exams.

B.1. Fall

Original comments

Some of these came by email; others are transcribed from exam papers. In either case, I do not make any stylistic corrections; mistakes can serve as a reminder that the students are not native English speakers. Nothing is left out; dots of ellipsis [. . .] are by the students.

Yunus

Lesson was generally nice. Although presentations of students are necessary, I think that you should talk more because you know the connections of propositions with other things. And learning these connections is very exciting.

For the exam, I think greek alphabet part is not necessary. I just memorized it and unfortunately I am sure that I will forget it.

Elif

First of all thank you that this course is opened. History and philosophy of mathematics are interesting and I am glad to take such kind of course from my own department this time. I like course material (conics is a bit difficult compared to elements but it is also good choice) and the connection with language add a variation. Presentations shows us our deficiencies, so they were very useful for us. Maybe it were possible that not to choose before the lesson the person who presents the proposition in order to make everyone prepared and have higher interaction level. Homeworks or quizzes about alternative proofs may force us to consider them more frequently. Exams were parallel to lessons and measure what we have learned. In short, I am pleased to have attended to this course.

Tolga

Firstly, I didn't suppose that this course might be exciting for me. I added this course after the add-drop week, so I couldn't attend the lesson until this week. We have studied 'The Elementa', Euclid, so I had a chance to study on this book. After all, this book helped me to learn how to examine on mathematics. secondly, this final exam is extraordinary. because of that I coudn't well done that I supposed but it is certain that this exam gave a chance to apply the propositions we learned. Moreover, the lessons were interactive so that it increases my attandences. Shortly, I think it is a good and exciting course for me. I am happy to take this course. Thanks for everything. Good holiday.

Besmir

I wish the course was a little more math history, that would be very nice I think. The exam maybe should be more alike the things we are usually used to in the texts we had. Considering the risk I now find my self in, maybe a second midterm would be good.

Ali

due to excessive amount of sidenotes, there is not much place left for us to write. it'd be proper to give those problems on a 5-page exam paper. also, size de iyi tatiller*

Tuğba

I had trouble in understanding the propositions of the last book we covered in class, Apollonius. It would be better if the books which are easier to understand and follow in lessons are covered

Rashad

I think exam and course were nice. *Maybe* a *little bit* more *history* will be good. I wish you to have a nice holiday, too.

Melis

The course can be improved by making it "more" history included and "less" mathematics included. Maybe other than learning history of mathematical concepts, we also need to learn the lives of the mathematicians who discovered those mathematical concepts.

Seçil

Firstly it is good to learn about Euclid an Apollonius. We think (as a mathematics student) we know some geometry but we doesn't even know Euclid's Elements.During this semester if we read some articles or some books about Euclid and Apollonius lecture would be more educational.Sometimes classes were like a geometry class.I think we still don't understand why Euclid and Apollonius are so important in the history of mathematics.Moreover I think we should talk about science in Ancient Greek .

---

*Happy holidays to you too.

Secondly, final exam was a good exam .I just blame myself since I didn't read definitions carefully. It was a good lecture thank you for being so nice to us.

Have a nice holiday.

Mürsel

It is a good chance learning history of mathematics. In our country these terms are not investigated at high school. At this stage I think this will be better if we learn than before. I like this course but it can be improved by more modern terms. At class when you compare those think with later work, we enjoyed to much. Thank you!

Nur

The book Appollionus was really hard to understand, it can be reduced from the consept of lecture. Giving some exercise sheets for exams can help the students to guide how to work for exam. For example I try to memorize everything covered but still can not get a high grade :(

Özge

Before taking this course, I've heard something about it from my friends who took in the previous years. They were presenting the biographies of the ancient mathemecians and their famous theorems u.s.w. We had partially done the same by presenting Euclid's Elements and Apollonius' Conics but it would be nice if we learnt something about also their lives, how did they become who they are . . .

Tolgay

I wanted to write my opinions about the lecture. Sorry about late-sending this email. But you know . . . finals.

First of all, I really enjoyed the course. The thing is I am not really motivated for a big part of our courses. I like geometry, I like history so the idea of this lecture was good for me. But the "history" part was not that emphasized. I mean when we check our website of department

> MATH 303 History of Mathematical Concepts I (3-0)3 Mathematics in Egypt and Mesopotamia, Ionia and Pythagoreans, paradoxes of Zeno and the heroic age. Mathematical works of Plato, Aristotle, Euclid of Alexandria, Archimedes, Appolonius and Diophantus. Mathematics in China and India. Prerequisite: Consent of the instructor.

It seems its more historical. I think I prefer what we did but still . . .

Maybe there could be a part where the other matematicians' works are presented. I think the students of mathematics should know some basic history of maths but I am don't know if it's here to learn it. They can, if they're interested, easily read and make research about them.

And lastly, maybe a course website can be constructed by students as well . . .

Thank you.

Taner

The lecture is especially not boring as the other courses that I have taken from our department.You made us think about the proofs and formalize the mathematical stuff by using geometry(görsel kanıtlama yöntemleri in Turkish). But grading is more important for the students at METU so you can name the cost(howmany points we are supposed to take from a presentationproof) of the classes.I accept that this is not nice but you will see that we students will attempt the course more than we do now. Also if we learn the greek alphabet better this may benefit us.To give an example, I sometimes couldn't understand the proofs from the books that are in your site.Or may be some more books can be suggested for us to have a look for the proofs of the propositions. This is all that I can find for now.Thanks a lot for your help during the classes, and your understanding about being a teacher =) See you in the next semester hocam

My responses

Here I summarized some responses to the bonus problem on the last exam, and I added my comments. I did not finish or distribute this work; but students' comments did influence my writing of the course webpage for Math 304; see the beginning of Part II.

A. *There was not enough space on the exam paper.* Sorry!

B. *There were no problems on ellipses or hyperbolas on the exam. Exercise sheets would have been useful in preparing for exams.* In a course like calculus, the objective is to be able to solve problems, and the purpose of the textbook is to help you meet this objective. Such is probably not the purpose of Euclid or Apollonius.

Briefly, I see the objective of this course as to gain some insight into what mathematics *is.* Two millenia and a few centuries ago, Euclid, Archimedes, and Apollonius were doing something that we can recognize as mathematics; but is it really the same as what we call mathematics today? The only way I know to answer this is to read these mathematicians and try to understand what they were doing.

This course has exams because exams are a standard means today for assessing student progress. But my hope is that, if one does the readings for the course with sufficient . . .

C. *Learning the Greek alphabet was not necessary.* Indeed, I'm sure that most mathematicians outside of Greece cannot recite the Greek alphabet. Unfortunately most mathematicians have not read Euclid or Apollonius either. But mathematicians *use* every letter of the Greek alphabet (except perhaps ο), and they may have opinions about ancient Greek mathematics (such as "Euclid discovered the parallel postulate," or "Euclid invented the axiomatic method"). Such opinions really ought to be based on reading the original works. It would be best to read these works in the original Greek, since translations can introduce distortions. Indeed, as we discussed in class:

a) Euclid says "Let a straight line have been drawn," but the translator might say "Let a straight line be drawn";
b) Euclid says "Let the given straight line be AB," but the translator might say "Let AB be the given straight line."

These distinctions are subtle and are perhaps not mathematically important. Also, we cannot all learn ancient Greek. However, as mathematicians, we ought to be able to recognize words like κύκλος and παραβολή.

D. We should have done more *history.*

E. *Apollonius is difficult; students have trouble presenting some propositions, and this causes difficulties for others in the class.*

B.2. Spring

Salih A.

It will be more helpful for us, if you explain (teach) the course instead of the students. When we make presentations all of us know only their subject well, because we cannot concentrate on other students subjects. You can give homeworks or some other projects instead of teaching. Because listening subject from a lecturer or a student is very different. Thanks for everything . . .

Ece

In this course, if we want to solve questions we need to think and work on more about them. To make presentations is a good idea. At least some of the students get prepared the course and know the propositions or corrollaries . . .

I think, take home exams or, only homework questions without exams will be better idea. Because I think if the exams would be take home style, the students (we) meet all together and think together. In this way we all need to learn all the corollaries or etc., because in the questions I can use some of the properties and the other student can use other ones. So none of us can solve questions, but we all can solve some part of them. But if we do them together, we may solve the questions. As a old people say "Bir elin nesi var, iki elin sesi var."*

Duygu

Don't get angry but I totally think student presentations is not a good idea. Personally I like old-fashioned classroom style, the teacher lectures and kids listen and takes notes. If you try the classical method, I think both you and students will be happier. (and it would be better for exams too, it's good to have a proper notebook for the exams.) Also thank you for recommendation letters ☺

*"What has one hand got? Two hands have a sound."

Burhan

expecially, A book which has a fluently english and "güncel"[*] english words. Also at the class, you can be more active about teaching lesson because when students try to teach it can be difficult to understand. Also, we are not recognize ancient terms about mathematics so I think before these course, Math 303 need to be a prerequisite lesson.

Gökçen

This course wasn't that clear because of the language of the texts. It was a little bit strange and challenging to understand the content of them. Maybe due to this, I couldn't enjoy than I expected at the beginning of this semester . . . Because for me mathematics is getting much more enjoyable and attractive when I can understand and can do something about it. Only these two points bothered me during whole this semester . . . (Again thanks for your understanding . . .)

Seray

This semester I started working and this course was the only one requiring attendence among 3 courses I've taken. I could only attend this course at the beginning of the semester. Therefore I don't have a lot to say about improving the course. But I have some observations. First, the presentations weren't effective enough to get us to the level of being successful at your exams. Also written group homeworks would be more motivating than making presentations.

Besmir

I think the conic section questions are too confusing and I find it hard to see what is going on. Thank you.

---

[*] Current.

Ali

Due to lots of mathematicians being studied in the semester, students may grow tired, because each new mathematician requires a new, more or less, mindset than the previous one. So reducing the number of people studied may be a good idea. Also, if you're lecture this course next year, do put previous exams on the net so that students may see what your style is, what kind & type of questions you ask.

Oğuzhan

Well I believe that understanding this stuff is not the main trouble for us; however when it comes to applying to questions in the exam, we are a bit confused (at least me!!). So it may be quite useful if some applications of these propositions / lemmas are distributed to the class (similar to recitation hour!) I enjoyed attending this class. Thank you.

Yasemin

First of all, I thank you for your kindness and help during this semester. In order to improve the course, in my opinion it is better for the lectures to be guided by you more, instead of the students. For participation in class, student's proving the statements is good; however, it would be better for anyone to prove some important part from the lecture notes which is choosen by the student at the beginning of the semester, and then you may expect a more qualified presentation and proof, digged into the topic by the student itself, and this presentation might worth more credits for this course, such as % 25.

Also you may give some bonus tasks, since this course is not a simple one, then the catalogue grading would fit everyone. Have a nice holiday. Best regards.

Melis

This course is all about the history of mathematical concepts, but I would wish to learn more about the "people" who discovered those concepts. Surely, it's totally up to me to learn about them by myself, but I would prefer to be asked about the people rather than what they discovered. We, as the upcoming mathematicians, are supposed to know about the history of mathematics in all aspects. For instance, last semester, in Math 303 course, I was very glad to be taught the Greek alphabet although I can already speak Greek. It was a completely different perspective for both me and the rest of the class.

My answer to this question has become "my feedback to the course" more than "my suggestions for improving the course," but I'm finding it useful to transfer my ideas about the course. For one thing, I enjoyed attending the classes of Math 303 more. I could concentrate on it more. I don't think that this is about the easiness of Math 303. It's about that I liked the content of the course more. The reason why I took Math 304 without hesitate is exactly this. As you must have realized, I couldn't focus on Math 304 during the semester.

I hope my feedback gives you some idea about how this course made the class feel. Thank you.

Salih K.

I can not speak English very well. So I can not explain my sentences to teacher. This is my problem, I know. In my opinion, Cem TEZER must be the instructor. David Hoca is a good teacher except for me.

Zhala

My suggestions for improving the course are that:

Firstly, I think the content of the course is good but the system of learning is not. I mean, presentations handed by students should be more well-prepared. Visuals should be used. Moreover, Publishes of TÜBİTAK can be used in order to improve our analytic thinking on certain problems coming from history of mathematics. Also, grp presentations can be held and can make more understandable the content.

Thanks a lot.

Şule

(I think) Giving take home quizzes may help students to think on lecture materials. (To be familiar with problems etc.)

Mehmet Ş.

I think this course needs a book which is understood easier than the current textbook.

Makbule

The content of the course is becoming harder at the end of the term. Lemma's and theorems that we covered after midterm seem as high levelled mathematical concepts. Therefore, the course may be clearer when you tell the idea of the propositions, theorems etc. to us first. It is really convinient and efficient to go over the notes which we take during the class. I mean, it is easier to understand your own notes than the ones in the book.

The presentations make the course more interactive. I'm happy with the idea of presentations. But it may be more beneficial if you repeat the main points and the idea of the propositions after our presentations. I talked to almost everyone in the class about this and the general idea about presentations is parallel to mine.

The last thing I want to mention is that the lackness of the questions related to the topics we covered in the class. It is really difficult to handle with the questions for the first time during the exam. It can be better if you give some exercises before exams.

C. Collingwood on history

From the *Autobiography* [14]:

> I expressed this new conception of history in the phrase: 'all history is the history of thought.' You are thinking historically, I meant, when you say about anything, 'I see what the person who made this (wrote this, used this, designed this, &c.) was thinking.' Until you can say that, you may be trying to think historically, but you are not succeeding. And there is nothing except thought that can be the object of historical knowledge. Political history is the history of political thought: not 'political theory', but the thought which occupies the mind of a man engaged in political work: the formation of a policy, the planning of means to execute it, the attempt to carry it into effect, the discovery that others are hostile to it, the devising of ways to overcome their hostility, and so forth . . . Military history, again, is not a description of weary marches in heat or cold, or the thrills and chills of battle or the long agony of wounded men. It is a description of plans and counter-plans: of thinking about strategy and thinking about tactics, and in the last resort of what men in the ranks thought about the battle.
>
> On what conditions was it possible to know the history of a thought? First, the thought must be expressed: either in what we call language, or in one of the many other forms of

expressive activity . . . Secondly, the historian must be able to think over again for himself the thought whose expression he is trying to interpret . . . If some one, hereinafter called the mathematician, has written that twice two is four, and if some one else, hereinafter called the historian, wants to know what he was thinking when he made those marks on paper, the historian will never be able to answer this question unless he is mathematician enough to think exactly what the mathematician thought, and expressed by writing that twice two are four. When he interprets the marks on paper, and says, 'by these marks the mathematician meant that twice two are four', he is thinking simultaneously: (a) that twice two are four, (b) that the mathematician thought this, too; and (c) that he expressed this thought by making these marks on paper . . .

This gave me a second proposition: 'historical knowledge is the re-enactment in the historian's mind of the thought whose history he is studying.'

D. Departmental correspondence

Here are some emails about the course that were shared within the METU mathematics department.

D.1. Wednesday, April 28, at 13:01

I wrote to `odtu-math`*:*

Since it is time to make our course requests for next fall, I thought I would talk about what I have been doing this year with Math 303 and 304, 'History of Mathematical Concepts'. I would be happy to teach this course again next year; but I would also be happy if somebody else was interested in teaching the course as I have been.

I have three principles for the course:

1. Our only textbook is original sources (in translation as necessary): Euclid, Apollonius, al-Khwarizmi, Descartes, . . .
2. The 'teacher' does not lecture; the students present at the blackboard what they have read.
3. Attendence is required.

There are details on the web pages

`http://www.metu.edu.tr/~dpierce/Courses/303/`

`http://www.metu.edu.tr/~dpierce/Courses/304/`

In addition, I keep a journal of what goes on in class. My record of the first semester is 42 pages. If you want to see it, let me know.

We spent most of last semester reading Euclid. Many of the propositions were familiar to the students; but the students had not proved these propositions before. It seemed a shame that the students had had to wait till their third or fourth year at university to prove these propositions. Euclid's propositions were part of the basic education of most of the great mathematicians whose theorems we try to teach. Indeed, it might be good if Math 111 consisted (in part or whole) of reading and presenting Euclid. For example, **proportion** as Euclid defines it is an excellent example of an equivalence relation.

Meanwhile, Math 303 is a place where our students can read Euclid (and Apollonius, or Archimedes, or . . .). As I said, I am happy if either I or somebody else does this reading with them next year.

D.2. Wednesday, April 28, at 18:55

Sergey responded on `odtu-math`*:*

In my opinion, a few first lectures should be really devoted to mathematics of antique times and Middle Ages, but the most importand and most interesting events in mathematics happened later, and one should spare enough time to discuss the works of Newton, Euler, Lagrange, Galois, Abel, Gauss, Riemann, Klein, Hilbert, Poincare, and many other giants. One should discuss the history of geometry (famous old problems, non-Euclidean geometry, Italian Algebro-geometric school), of algebra (evolution of the concepts, like numbers, groups), of Analysis (the Newton-Leibnitz dispute, the problem of foundations, notorious mistaken 'theorems', fake references like 'L'Hospital rule', etc.). It is good to say about the history of the first Math journals, Academies of Science, about their Competitions and Awards. One should certainly discuss Hilbert's problems, the history of Fields medals, solution of the most outstanding problems (of Fermat, of Poincare, Four-colour, etc.), some new theories and trends in Math in the last century, and may be stop with the Millenium problems.

I can give such a course myself, or welcome anybody else who would do it!

D.3. Thursday, April 29, at 11:44

I responded:

Sergey wrote:

> In my opinion, a few first lectures should be really devoted to mathematics of antique times and Middle Ages, but the most importand and most interesting events in mathematics happened later, and one should spare enough time to discuss the works of Newton, Euler, Lagrange, Galois, Abel, Gauss, Riemann, Klein, Hilbert, Poincare, and many other giants.

Thanks for writing. However, I don't know what your point is. You refer to 'a first few lectures'. A few first lectures of **what?** You are replying to my email about Math 303/4, so maybe you are referring to lectures in this course. However, I wrote:

> I have three principles for the course: . . .
>
> 2. The 'teacher' does not lecture; the students present at the blackboard what they have read.

So there are no lectures in my course. Or rather, everybody in the classroom is a lecturer. You don't seem to address this point. But my undergraduate education consisted entirely of classes like this. I was very happy with the arrangement, and I decided to see if it would work at METU. I believe I have had some success.

However, we are reading Newton's *Principia* now. I don't know if we shall have time for anything else. Recently a young woman whom I'll call 'Yolanda' [Yasemin] was presenting Lemma VII [see p. 178 above], which you can see at:

`http://en.wikisource.org/wiki/The_Mathematical_Principles_of_Natural_Philosophy_(1729)/Book_1/Section_1#Lem7`

When 'Yolanda' got to Corollary 2, she said and wrote that AD, DE, BF, FG, AB, and the arc ACB had [ultimately] the ratio of equality.

I said I didn't believe it. In fact she had miscopied Newton. I hoped she would try to work out a proof and see her mistake. But she only beamed at me and said, 'It's hard to believe, but true!'

I finally asked 'Yolanda' to check her text. She saw that she should have had AE for DE, and BG for FG. But she couldn't give a proof of the correct statement. She just muttered something about how Newton was smarter than she was.

I went to the board and suggested a proof. One of the most interested and active students in the class, 'Oscar' [Oğuzhan], was skeptical; but when you are talking (for the first time in history, perhaps) about the ratios with which quantities vanish, skepticism is to be expected. 'Sara' [Şule] seemed to think at first that Lemma VII followed immediately from Lemma VI.

And so the discussion continued. Thus a number of students became collectively engaged in puzzling out what Newton was talking about. Unfortunately it doesn't happen much in my class. Students come to class and present the propositions assigned to them, but often they haven't really understood the point of the propositions, or their proof. In their presentations, they may say, 'he says this, then he says that', rather than saying **we** have this, and **therefore** we have that.

How can they do anything else? One difficulty is that students are taking several other courses, in particular math courses, which also demand their attention. Of greater concern is that students are trained to believe that books and teachers are unquestionable authorities. I hope to encourage them to see things differently. This is the main point.

By the way, reading Cardano's *Ars Magna* in my course was perhaps useful for this purpose. I hadn't read Cardano before, but I thought that, in Math 304, we might read his solution of cubic equations. Then I found more and more of Cardano that seemed worth reading in class.

After reading more carefully with the students, I had to conclude that either Cardano was a bad writer, or else he really didn't understand what he was doing. He also makes computational mistakes, which students discovered.

Unfortunately the only available English translation of Cardano is unsatisfactory, because it uses modern algebraic notation. Therefore I also gave the students the original Latin to look at. I think there is no point to studying pre-Cartesian mathematics unless one tries to forget about our modern symbolic tools. Descartes thought the Ancients really had such tools too; but this is not at all clear.

Sergey, let me repeat what you said:

> In my opinion, a few first lectures should be really devoted to mathematics of antique times and Middle Ages, but the most importand and most interesting events in mathematics happened later, and one should spare enough time to discuss the works of Newton, Euler, Lagrange, Galois, Abel, Gauss, Riemann, Klein, Hilbert, Poincare, and many other giants.

The most important and most interesting events happened later? This makes as much sense as saying that *War and Peace* is more important and more interesting than the *Iliad* and the *Odyssey.* But have you **read** Newton, Euler, Lagrange, and the others you mention? Do you propose to read them in class **with students?**

For Math 304, I wondered if we could use something like Struik's *Source Book in Mathematics, 1200–1800.* I decided against it. There is not much point in reading the short passages provided by Struik, just so that one can say, 'I've read Leibniz' or 'I've read Bernoulli'. A writer worth reading is worth spending time with, over the course of many pages.

Struik gives a passage from Cardan with a solution of what we write as

$$x^3 + 6x = 20. \qquad (*)$$

A third part of each of Struik's pages is filled with footnotes explaining what Cardan is doing. Only Cardan is **not** doing what is in the footnotes. As Struik shows in his notes, **we** can solve the cubic equation

$$x^3 + px = q$$

by substituting

$$x = u - v$$

and solving first for u and v. But either Cardan doesn't really see this himself, or else he is hiding it. Cardan gives a formula for x, and he can prove it is correct by substitution; but he shows no interest in **deriving** the formula. Struik does not address this point.

Neither does Boyer, whose text has (I believe) been traditionally used for Math 303/4. In his section on Cardan's solution, Boyer just writes,

> The solution of this equation covers a couple of pages of rhetoric that we should now put in symbols as follows: Substitute $u - v = x$ . . .

In that 'rhetoric', Cardan shows in effect that, **if** we have the simultaneous equations

$$u^3 - v^3 = 20, \qquad uv = 2,$$

then $x = u - v$ in $(*)$ above. He doesn't say **why** we should start with those simultaneous equations. Neither does he explain how to solve those simultaneous equations: he just tells you the formula for the solution. Actually this seems to be just what mathematics is in the minds of (some of) our students, who love formulas, no matter where they come from, as long as they can be used on the exam. But I blame the university entrance exam system for this (and perhaps teachers who are earlier products of this system).

> One should discuss the history of geometry (famous old problems, non-Euclidean geometry, Italian Algebro-geometric school), of algebra (evolution of the concepts, like numbers, groups), of Analysis (the Newton-Leibnitz dispute, the problem of foundations, notorious mistaken 'theorems', fake references like 'L'Hospital rule', etc.). It is good to say about the history of the first Math journals, Academies of Science, about their Competitions and Awards. One should certainly discuss Hilbert's problems, the history of Fields medals, solution of the most outstanding problems (of Fermat, of Poincare,

> Four-colour, etc.), some new theories and trends in Math in the last century, and may be stop with the Millenium problems.
>
> I can give such a course myself, or welcome anybody else who would do it!

Well Sergey, there is a procedure for opening new courses. Or if you mean to be describing how **Math 303/4** should be taught, then please say so. You seem to be describing a lecture course; if so, it is not a course that I would consider myself competent to teach.

Lecturing **mathematics** is fine, since the listeners can check the lecturer's claims by using the critical powers of their own reason. Again though, I am sorry that even some students in Math 303/4 don't use these critical powers very much. In any case, lecturing about what happened in the past is a different matter. For example, perhaps we all grew up with the idea that there was a crisis in ancient mathematics owing to the discovery of incommensurable magnitudes. We may tell students about this if we happen to prove to them the irrationality of the square root of 2. However, it seems there is simply no evidence of an ancient crisis.

Of course events of more recent centuries may be better documented.

D.4. Friday, April 30, at 12:27

Sergey wrote again to `odtu-math`*:*

It would be interesting to discuss your ideas: I would prefer to do it privately, not involving people not interested in this subject. I will just try now to state clearly my opinion which differs from yours: a course ‘History of Math concepts’ is really needed for our Department just because it helps to understand better mathematics. There are impotant topics to be covered, and they should not be missed. For undergraduate students, an idea to replace a lecture course by a seminar course does not seem good: stdents may really have more fun (like in a course of singing, or dancing), and even study a few selected topics better, but overall they will be far behind the syllabus. A kind of a seminar that you proposed instead of lectures would be perfect for graduate students in History Department, who really need to learn how to work with the original sources.

E. Notes on Greek mathematics

[I put these notes on the Math 303 webpage at the beginning of the semester.]

E.1. Introduction

Some time in the 3rd century B.C.E., Apollonius of Perga wrote eight books on **conic sections.** We have the first four books [4, 5] in the original Greek; the next three books survive in Arabic translation [3]; the eighth book is lost. As Apollonius tells us in an introductory letter, his first four books are part of an elementary course on the conic sections.

Before Apollonius, around 300 B.C.E., Euclid published the thirteen books of the *Elements* [21, 23, 25], a work of mathematics of which some parts could well be used as a textbook today. The *Elements* provide a good example of mathematical exposition and of what it means to prove something.

In 2008, getting ready to teach a course on the conic sections,* I wrote some notes on ancient mathematics. Using those notes, I have prepared the present notes, for use in a course called 'History of Mathematical Concepts I' at METU—a course in which participants will read Euclid and Apollonius.

*At the Nesin Mathematics Village, Şirince, Selçuk, İzmir, Turkey.

In the latter sections of these notes, I look at some general features of ancient mathematics as I understand it. Meanwhile, in §E.3, I jump forward in history to Descartes, to see the sorts of improvements that he thought he was making to the mathematical practice of mathematicians like Euclid and Apollonius.

Because I shall occasionally refer to some Greek words, I review the Greek alphabet in [Appendix F.]

E.2. Why read the Ancients?

As an undergraduate, I attended a college* where Euclid and Apollonius were used as textbooks. They were so used, I think, not because they were considered to be the *best* textbooks, but because they *had been* textbooks for countless generations of mathematicians: therefore (the idea was), one might gain some understanding of humanity and oneself by reading these books. (The same is true for Homer, Aeschylus, Plato, and the other great books read at the college.)

Now, having become a professional mathematician, I ask what Euclid and Apollonius have to offer the mathematician of today. It is in pursuit of an answer to this question that I prepare these notes—which therefore are part of an ongoing project.

I prepare these notes also for the sake of honesty about what students are asked to learn. The curves called **conic sections** are a standard part of an elementary course of mathematics. The origin of such curves is in the name: they are obtained by slicing a cone. Apollonius treated the curves in this way. But

*St John's College, with campuses in Annapolis, Maryland, and Santa Fe, New Mexico, USA [see p. 11].

in math courses today, the conic sections are usually given as the curves defined by certain equations, such as

$$ay = x^2 \qquad \text{or} \qquad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

Or perhaps the curves are given in terms of foci and directrices. A textbook may *assert* that the curves so defined can indeed by obtained as sections of cones; but it is rare that this assertion is justified.

One calculus textbook* writes:

> In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections,** or **conics,** because they result from intersecting a cone with a plane as shown in Figure 1.

(I omit the author's figure.) The conic sections result from intersecting a cone with a plane: this can be understood as a *definition* of the conic sections. Let us call it Definition I. More precisely, this definition distinguishes three kinds of conic sections, depending on the angle of the plane with respect to the cone. One kind of conic section is called the *parabola,* and the text continues under the heading *Parabolas:*

> A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**) . . . In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges . . . We obtain

*James Stewart, *Calculus,* fifth edition, p. 720. This text is currently in use at METU.

> a particularly simple equation for a parabola if we place its vertex at the origin O . . .

Here then is another definition of the parabola; call it Definition II. Definitions I and II are equivalent in that they define the same objects; but the author does not clearly say so, much less prove it. I don't think he *needs* to prove the equivalence; but at least he ought to state that he is not going to prove it.

Perhaps the author expects the reader to *infer* the equivalence of Definitions I and II. But this is not his style. He is usually eager to give his readers every assistance. Note for example that he apparently does not trust readers to infer for themselves that parabolas are worth studying. Before concluding anything from his definition of parabolas, the author feels the need to tell the reader how *useful* parabolas are.

Another textbook* follows a similar procedure, first defining the conic sections *as such,* then defining them in terms of foci and directrices. Between the two definitions, the writer observes that the intersection of a cone and a plane will be given by a second-degree equation. This suggests that the quadratic equations to be derived presently in the book may indeed define conic sections. However, no attempt is made to prove that every curve defined by a quadratic equation can be obtained as the section of a cone. The author observes:

> After straight lines the conic sections are the simplest of plane curves. They have many properties that make them useful in applications of mathematics; that is why we include a discussion of them here. Much of this material is optional from the point of view of a calculus course, but familiarity with the properties of conics can be very important in

*Robert A. Adams, *Calculus: a complete course,* fourth edition, p. 476. This text was formerly used at METU.

> some applications. Most of the properties of conics were discovered by the Greek geometer, Apollonius of Perga, about 200 BC. It is remarkable that he was able to obtain these properties using only the techniques of classical Euclidean geometry; today most of these properties are expressed more conveniently using analytic geometry and specific coordinate systems.

Again, the justification offered for the study of the conic sections is their usefulness. But as for 'expressing' the properties of conic sections, which of the following expresses better what a conic section *is?*

1. It is the intersection of a cone and a plane.
2. It is the intersection of the surfaces defined by the equations

$$ax + by + cz + d = 0, \qquad (x - ez)^2 + y^2 = fz^2.$$

What the author means, I think, is that it is convenient to define certain curves 'analytically'—that is, in a coordinate system such as Descartes introduced; properties of the curves can then be obtained by further analysis. But showing that those curves are conic sections is a whole other problem, not addressed in the book.

By the way, despite what the last quotation suggests, I am not sure that obtaining nice results with limited mathematical tools is remarkable in itself. The tools of an artisan depend on what is available in the physical environment; but the tools of a mathematician depend only on imagination. A mathematician without the imagination to come up with the best tool for the job would seem to be an unremarkable mathematician.

The first chapter of Hilbert and Cohn-Vossen's *Geometry and the Imagination* [31] contains a beautiful account of how various properties of the conic sections arise from consideration of the cones from which the sections are obtained. However, the cones considered by the authors are all *right* cones. Apollonius does not make this restriction. Hilbert and Cohn-Vossen give an etymology for the names of the ellipse, the hyperbola, and the parabola: it involves eccentricity. The etymology is plausible, but it appears to be literally incorrect, as a reading of Book I of Apollonius would show.

Mathematics reveals underlying correspondences between seemingly dissimilar things. Sometimes we treat these correspondences as identities. This can be a mistake. There is a correspondence between conic sections and quadratic equations. But are the sections *really* the equations? One cannot answer the question without considering *conic sections as such,* as Apollonius considered them.

E.3. Synthesis and analysis

It may be said that, in reading Euclid and Apollonius, we are going to do **pre-Cartesian** mathematics: mathematics as done before (well before) the time of René Descartes (1596–1650).

The geometry pioneered by René Descartes is called **analytic geometry;** by contrast, the geometry of ancient mathematicians like Euclid and Apollonius is sometimes called **synthetic geometry.** But what does this *mean?* The word *synthetic* comes from the Greek συνθετικός, meaning *skilled in putting together* or *constructive.* This Greek adjective derives from the verb συντίθημι *put together, construct* (from συν *together* and τίθημι *put*). The word *analytic* is the English form of ἀναλυτικός, which derives from the verb ἀναλύω *undo, set free, dissolve* (from ἀνα *up,* λύω *loose*). Although we refer to ancient geometry as synthetic, the Ancients evidently recognize both analytic and synthetic methods. Around 320 C.E., Pappus of Alexandria writes [53, p. 597]:

> Now **analysis** (ἀνάλυσις) is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a *reverse solution* (ἀνάπαλιν λύσις).
>
> But in **synthesis** (συνθέσις), proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.
>
> Now analysis is of two kinds, one, whose object is to seek the truth, being called **theoretical** (θεωρητικός), and the other, whose object is to find something set for finding, being called **problematical** (προβληματικός).

This passage is not very useful without examples: I shall propose one presently. Meanwhile, I note that Pappus elsewhere [53, pp. 564–567] says more about the distinction between theorems and problems:

> Those who favor a more technical terminology in geometrical research use **problem** (πρόβλημα) to mean a [proposition*] in which it is proposed to do or construct [something];

*Ivor Thomas [53, p. 567] uses *inquiry* here in his translation; but

> and **theorem** ($\theta\epsilon\acute{\omega}\rho\eta\mu\alpha$), a [proposition] in which the consequences and necessary implications of certain hypotheses are investigated; but among the ancients some described them all as problems, some as theorems.

What really distinguishes Cartesian geometry from what came before is perhaps suggested by the first sentence of Descartes's *Geometry* [18, p. 2]:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.

From a straight line, Descartes abstracts something called *length.* A length is something that we might today call a positive real number.

Descartes takes the edifice of geometry that has been built up or 'synthesized' over the centuries, and reduces or 'analyzes' its study into the manipulation of numbers. To be more precise, he 'takes that which is sought as though it were admitted' in the following way. In Figure E.1, straight lines BE, DR, and FS are given in position (meaning their endpoints themselves are not fixed); and the sizes of angles ABC, ADC, and CFE are given. It is required to find the point C so that the rectangle with sides BC and CD has a given ratio to the square on CF. (This is a simplified version of the problem that Descartes takes up in the *Geometry.*)

In his analytic approach, Descartes assumes that C has already been found, as in the figure. We denote AB by x, and BC by y. The ratio $AB : BR$ is given; call it $z : b$. Then

$$RB = \frac{bx}{z}, \qquad CR = y + \frac{bx}{z} = \frac{zy + bx}{z}.$$

there is *no* word in the Greek original corresponding to this or to *proposition.*

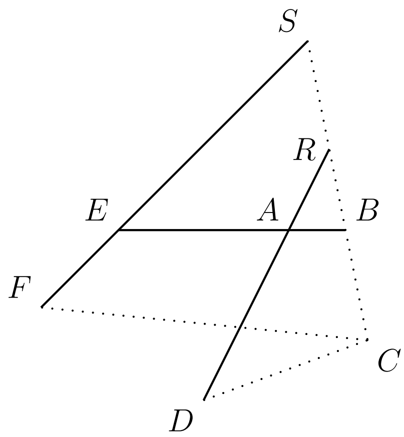

Figure E.1. Descartes's diagram

But $CR : CD$ is given; call it $z : c$. Then

$$CD = \frac{czy + bcx}{z^2}.$$

Also AE is given; call it k. And let $BE : BS = z : d$. Then

$$BE = k + x, \quad BS = \frac{dk + dx}{z}, \quad CS = \frac{zy + dk + dx}{z}.$$

Finally, if $CS : CF = z : e$, then

$$CF = \frac{ezy + dek + dex}{z^2}.$$

So it is given that the ratio

$$y \cdot \frac{czy + bcx}{z^2} : \left(\frac{ezy + dek + dex}{z^2}\right)^2$$

is constant. This gives us a quadratic equation in the unknowns x and y.

Descartes's method does not use explicitly drawn *axes* with respect to which x and y are measured. Also, the straight lines called x and y are not required to be perpendicular: they are merely not parallel.

Through analysis, we have found an equation that determines the point C. Since the equation is quadratic, the point C lies on (a curve that turns out to be) a conic section. When there are more straight lines in the problem, then the resulting equation may have a higher degree.

We do not get any sense here for what the curve of C *looks like.* We might get some sense by analyzing the equation for C. Apollonius will give us a sense for what conic sections look like by showing *how they are related to the cones that they come from.*

E.4. Theorems and problems

The text of Apollonius as we have it consists almost entirely of theorems and problems (in the sense of the last section). There are some introductory remarks, some definitions, but nothing else. The theorems and problems can be analyzed in a way described by Proclus,* in the fifth century C.E., in his commentaries on Euclid [43, p. 159]:

> Every problem and every theorem that is furnished with all its parts should contain the following elements: an *enunciation* (*πρότασις*), an *exposition* (*ἔκθεσις*), a *specification* (διορισμός), a *construction* (*κατασκευή*), a *proof* (ἀπόδειξις), and

*Proclus was born in Byzantium (that is, Constantinople, now İstanbul), but his parents were from Lycia (Likya), and he was educated first in Xanthus. He moved to Alexandria, then Athens, to study philosophy [43, p. xxxix].

a *conclusion* (συμπέρασμα). Of these, the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.

So many are the parts of a problem or a theorem. The most essential ones, and those which are always present, are enunciation, proof, and conclusion.

Alternative translations are: for ἔκθεσις, *setting out,* and for διορισμός, *definition of goal* [39, p. 10].

For an illustration, we may analyze Proposition 1 of Book I of Euclid's *Elements* (in Fitzpatrick's translation [25]). The proposition is a *problem:*

Enunciation. *To construct an equilateral triangle on a given finite straight-line.*

Exposition. Let AB be the given finite straight-line.

Specification. So it is required to construct an equilateral triangle on the straight-line AB.

Construction. Let the circle BCD with center A and radius AB have been drawn, and again let the circle ACE with center B and radius BA have been drawn. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively).

Proof. And since the point A is the center of the circle CDB, AC is equal to AB. Again, since the point B is the center of the circle CAE, BC is equal to BA. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another. □

Conclusion. Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

E.5. Conversational implicature

One apparent difference between the ancient and modern approaches to mathematics may result from a modern habit that is exemplified in a Russian textbook of the Soviet period [19, pp. 9 f.]:

> The student of mathematics must at all times have a clear-cut understanding of all fundamental mathematical concepts . . . The student will also recall the signs of weak inequalities: $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). The student usually finds no difficulty when using them in formal transformations, but examinations have shown that many students do not fully comprehend their meaning.
>
> To illustrate, a frequent answer to: "*Is the inequality* $2 \leqslant 3$ *true?*" is "No, since the number 2 is less than 3." Or, say, "*Is the inequality* $3 \leqslant 3$ *true?*" the answer is often "No, since 3 is equal to 3." Nevertheless, students who answer in this fashion are often found to write the result of a problem as $x \leqslant 3$. Yet their understanding of the sign $\leqslant$ between concrete numbers signifies that not a single specific number

can be substituted in place of x in the inequality $x \leqslant 3$, which is to say that the sign $\leqslant$ cannot be used to relate any numbers whatsoever.

The students referred to, who will not allow that $2 \leqslant 3$, are following a habit of ordinary language, whereby the *whole* truth must be told. According to this habit, one does not say $2 \leqslant 3$, because one can make a stronger, more informative statement, namely $2 < 3$. This habit would appear to be an instance of *conversational implicature:* this is the ability of people to convey or *implicate* statements that are not logically *implied* by their words [33, ch. 1, §5, pp. 36–40]. In saying *A or B [is true],* one usually 'implicates' that one does not know *which* is true.

This habit of implicature may be reflected in the ancient understanding, according to which *one* (ἕν) is not a *number* (ἀριθμός). In Book VII of the *Elements,* Euclid somewhat obscurely defines a **unit** (μονάς) as that by virtue of which each being is called 'one'. (This English version of the definition is based on the Greek text supplied in [21, Vol. 2, p.279].) Then a **number** is defined as a *multitude* (πλῆθος) composed of units. In particular, a unit is not a number, because it is not a multitude: it is one. Euclid does not bother to state explicitly this distinction between units and numbers, but it can be inferred, for example, from his presentation of what we now call the Euclidean algorithm. Proposition VII.1 of the *Elements* involves a pair of numbers such that the algorithm, when applied to them, yields a unit (μονάς). Then this unit is *not* considered as a greatest common divisor of the numbers; the numbers do not *have* a greatest common divisor; the numbers are simply relatively prime. If the numbers are *not* relatively prime, then the *same* algorithm yields their greatest common divisor. This observation appears to be the contrapositive of the first, but Euclid distinguishes it as Proposition VII.2 of the *Elements.*

Conversational implicature may be seen in Apollonius's treating of the circle as different from an ellipse.

E.6. Lines

In the old understanding, a **line** need not be straight. A line may have endpoints, or it may be, for example, the circumference of a *circle.* Indeed, according to the definition in Euclid's *Elements,*

> A **circle** (κύκλος) is a plane figure contained by one line (γραμμή) such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

A *straight* line (εὐθεῖα γραμμή) does have endpoints; but the straight line may be *produced* (extended) beyond these endpoints, as far as desired.

F. The Greek Alphabet

I have heard a rumor (see p. 48) that students can improve their mathematics simply by learning this alphabet.

A α	**a**lpha	I ι	**i**ota	P ρ	**rh**o
B β	**b**eta	K κ	**k**appa	Σ σ,ς	**s**igma
Γ γ	**g**amma	Λ λ	**l**ambda	T τ	**t**au
Δ δ	**d**elta	M μ	**mu**	Υ υ	**u**psilon
E ϵ	**e**psilon	N ν	**nu**	Φ ϕ	**ph**i
Z ζ	**z**eta	Ξ ξ	**x**i	X χ	**ch**i
H η	**ē**ta	O o	**o**micron	Ψ ψ	**ps**i
Θ θ	**th**eta	Π π	**p**i	Ω ω	**ō**mega

Figure F.1. The Greek alphabet

The first letter or two of the (Latin) name for a Greek letter provides a transliteration for that letter. However, upsilon is also transliterated by *y.* The diphthong $\alpha\iota$ often comes into English (*via* Latin) as *ae,* while οι may come as *oe.* The second form of the small sigma is used at the ends of words. In texts, the rough-breathing mark (῾) over an initial vowel (or ρ) is transcribed as a preceeding (or following) h (as in ὁ ῥόμβος *ho rhombos* 'the rhombus'). The smooth-breathing mark (᾿) and the three tonal accents (ά, ᾶ, ὰ) can be ignored. Especially in the dative case (the Turkish -e hali), some long vowels may be given the iota subscript (ᾳ, ῃ, ῳ), representing what was once a following iota ($\alpha\iota$, $\eta\iota$, $\omega\iota$).

Bibliography

[1] Apollonius of Perga. *Apollonius of Perga: Treatise on Conic Sections*. University Press, Cambridge, UK, 1896. Edited by T. L. Heath in modern notation, with introductions including an essay on the earlier history of the subject.

[2] Apollonius of Perga. *On conic sections*. Great Books of the Western World, no. 11. Encyclopaedia Britannica, Chicago, London, Toronto, 1952.

[3] Apollonius of Perga. *Conics. Books V–VII*, volume 9 of *Sources in the History of Mathematics and Physical Sciences*. Springer-Verlag, New York, 1990. The Arabic translation of the lost Greek original in the version of the Banū Mūsā, Volume I: Introduction, text and translation, Volume II: Commentary, figures and indexes, Edited with translation and commentary by G. J. Toomer.

[4] Apollonius of Perga. *Conics. Books I–III*. Green Lion Press, Santa Fe, NM, revised edition, 1998. Translated and with a note and an appendix by R. Catesby Taliaferro, with a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue, edited by Densmore.

[5] Apollonius of Perga. *Conics. Book IV*. Green Lion Press, Santa Fe, N.M., 2002. First English edition. Translation, introduction, and diagrams by Michael N. Fried.

[6] Archimedes. *The Two Books* On the Sphere and the Cylinder, volume I of *The Works of Archimedes*. Cambridge University Press, Cambridge, 2004. Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams, by Reviel Netz.

[7] Aristoteles. *Metafizik*. Sosyal Yayınlar, Cağaloğlu–İstanbul, 1996. Second printing. Turkish translation by Ahmet Arslan.

[8] V. I. Arnol′d. On the teaching of mathematics. *Uspekhi Mat. Nauk*, 53(1(319)):229–234, 1998.

[9] Benno Artmann. *Euclid—the creation of mathematics*. Springer-Verlag, New York, 1999.

[10] Robert S. Bart. *Notes to Accompany the Reading of Newton's* Principia Mathematica. St. John's College, Annapolis, Maryland, May 1968.

[11] Carl B. Boyer. *A history of mathematics*. John Wiley, New York, second edition, 1989. Edited and with a preface by Uta C. Merzbach.

[12] Girolamo Cardano. *The great art or the rules of algebra*. Translated and edited by T. Richard Witmer, with a foreword by Oystein Ore. The M.I.T. Press, Cambridge, Mass.-London, 1968.

[13] Paul J. Cohen. *Set theory and the continuum hypothesis*. W. A. Benjamin, New York–Amsterdam, 1966.

[14] R. G. Collingwood. *An Autobiography*. Clarendon Press, Oxford, 1978. First published 1939. With a new introduction by Stephen Toulmin. Reprinted 2002.

[15] R. G. Collingwood. *The Principles of History and other writings in philosophy of history*. Oxford, 2001. Edited and with an introduction by W. H. Dray and W. J. van der Dussen.

[16] Richard Dedekind. *Essays on the Theory of Numbers. I: Continuity and Irrational Numbers. II: The Nature and Meaning of Numbers.* Authorized translation by Wooster Woodruff Beman. Dover Publications, New York, 1963.

[17] Dana Densmore. *Newton's* Principia*: The Central Argument.* Green Lion Press, Santa Fe, New Mexico, third edition, 2003. Translations and Diagrams by William H. Donahue. Completely revised and redesigned. New sections added.

[18] René Descartes. *The Geometry of René Descartes.* Dover Publications, New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[19] G. Dorofeev, M. Potapov, and N. Rozov. *Elementary Mathematics: Selected Topics and Problem Solving.* Mir Publishers, Moscow, 1973. Translated from the Russian by George Yankovsky.

[20] Apostolos Doxiadis and Christos H. Papadimitriou. *Logicomix.* Bloomsbury, London, 2009.

[21] Euclid. *The Thirteen Books of Euclid's* Elements. Dover Publications, New York, 1956. Translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. In three volumes. Republication of the second edition of 1925. First edition 1908.

[22] Euclid. *The Bones: A handy where-to-find-it pocket reference companion to Euclid's* Elements. Green Lion Press, Santa Fe, NM, 2002. Conceived, designed, and edited by Dana Densmore.

[23] Euclid. *Euclid's Elements.* Green Lion Press, Santa Fe, NM, 2002. All thirteen books complete in one volume. The Thomas L. Heath translation, edited by Dana Densmore.

[24] Euclid. *Euclid's* Data, volume 45 of *Acta Historica Scientiarum Naturalium et Medicinalium Edidit Biblioteca Universitatis Hauniensis.* Museum Tusculanum Press. University of Copenhagen, Copenhagen, 2003. Or the importance of being given, The Greek text translated and explained by Christian Marinus Taisbak.

[25] Euclid. *Euclid's Elements of Geometry.* Published by the editor, revised and corrected edition, 2008. Edited, and provided with a modern English translation, by Richard Fitzpatrick, `http://farside.ph.utexas.edu/euclid.html`.

[26] Carl Faulmann. *Yazı Kitabı.* Türkiye İş Bankası Kültür Yayınları, 2001.

[27] Carl Friedrich Gauss. *Disquisitiones Arithmeticae.* Springer-Verlag, New York, 1986. Translated into English by Arthur A. Clarke, revised by William C. Waterhouse.

[28] S. H. Gould. *Russian for the mathematician.* Springer-Verlag, New York, 1972.

[29] Timothy Gowers, June Barrow-Green, and Imre Leader, editors. *The Princeton Companion to Mathematics.* Princeton University Press, Princeton, NJ, 2008.

[30] Paul Harvey, editor. *The Oxford Companion to Classical Literature.* Clarendon Press, Oxford, 1937. reprinted with corrections, 1980.

[31] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination.* Chelsea Publishing, New York, 1952. Translated by P. Neményi.

[32] David Hilbert. *The Foundations of Geometry.* Authorized translation by E. J. Townsend. Reprint edition. The Open Court Publishing Co., La Salle, Ill., 1959. Based on lectures

1898–99. Translation copyrighted 1902. Project Gutenberg edition released December 23, 2005 (`www.gutenberg.net`).

[33] Rodney Huddleston and Geoffrey K. Pullum. *The Cambridge Grammar of the English Language*. Cambridge University Press, Cambridge, UK, 2002. reprinted 2003.

[34] H. I. Karakaş. *Analytic Geometry*. M ⊕ V [Matematik Vakfı], [Ankara], n.d. [1994].

[35] Victor J. Katz, editor. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton University Press, Princeton and Oxford, 2007.

[36] Jacob Klein. *Greek mathematical thought and the origin of algebra*. Dover Publications, New York, 1992. Translated from the German and with notes by Eva Brann, Reprint of the 1968 English translation.

[37] Morris Kline. *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, New York, 1972.

[38] Barry Mazur. When is one thing equal to some other thing? In *Proof and other dilemmas*, MAA Spectrum, pages 221–241. Math. Assoc. America, Washington, DC, 2008.

[39] Reviel Netz. *The Shaping of Deduction in Greek Mathematics*, volume 51 of *Ideas in Context*. Cambridge University Press, Cambridge, 1999. A study in cognitive history.

[40] Isaac Newton. *The* Principia*: Mathematical Principles of Natural Philosophy*. University of California Press, Berkeley, CA, 1999. A new translation by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz. Preceded by "A guide to Newton's *Principia*" by Cohen.

[41] David Pierce. Model-theory of vector-spaces over unspecified fields. *Arch. Math. Logic*, 48(5):421–436, 2009.

[42] Plato. *Republic.* Number 237, 276 in the Loeb Classical Library. Harvard University Press and William Heinemann, Cambridge MA and London, 1982, 1980. with an English Translation by Paul Shorey, in two volumes, first printed 1930 and 1935.

[43] Proclus. *A Commentary on the First Book of Euclid's* Elements. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1992. Translated from the Greek and with an introduction and notes by Glenn R. Morrow. Reprint of the 1970 edition. With a foreword by Ian Mueller.

[44] Abraham Robinson. *Non-standard analysis.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1996. Reprint of the second (1974) edition. With a foreword by Wilhelmus A. J. Luxemburg. First edition 1965.

[45] Lucio Russo. *The Forgotten Revolution: How Science Was Born in 300* BC *and Why It Had to Be Reborn.* Springer-Verlag, Berlin, 2004. Translated from the 1996 Italian original by Silvio Levy.

[46] David Eugene Smith. *A Source Book in Mathematics.* 2 vols. Dover Publications, New York, 1959. Unabridged republication of the first edition, 1929, published by McGraw-Hill.

[47] Raymond Smullyan. *What is the Name of This Book?* Prentice-Hall, Englewood Cliffs, New Jersey, 1978. The Riddle of Dracula and Other Logical Puzzles.

[48] Herbert Weir Smyth. *Greek Grammar.* Harvard University Press, Cambridge, Massachussets, 1980. Revised by Gordon M. Messing, 1956. Eleventh Printing. Original edition, 1920.

[49] Michael Spivak. *A comprehensive introduction to differential geometry. Vol. I.* Publish or Perish, Wilmington, Del., second edition, 1979.

[50] D. J. Struik, editor. *A Source Book in Mathematics, 1200–1800.* Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1986. Reprint of the 1969 edition.

[51] Dirk J. Struik. *A Concise History of Modern Mathematics.* Dover, New York, fourth revised edition, 1987.

[52] Ivor Thomas, editor. *Selections Illustrating the History of Greek Mathematics. Vol. I. From Thales to Euclid.* Number 335 in Loeb Classical Library. Harvard University Press, Cambridge, Mass., 1951. With an English translation by the editor.

[53] Ivor Thomas, editor. *Selections Illustrating the History of Greek Mathematics. Vol. II. From Aristarchus to Pappus.* Number 362 in Loeb Classical Library. Harvard University Press, Cambridge, Mass, 1951. With an English translation by the editor.

[54] Bilge Umar. *Türkiye'deki Tarihsel Adlar [Historical Names in Turkey].* İnkilâp, İstanbul, 1993.

[55] Jean van Heijenoort, editor. *From Frege to Gödel: A source book in mathematical logic, 1879–1931.* Harvard University Press, Cambridge, MA, 2002.

[56] François Viète. *The analytic art.* Dover Publications, Mineola, NY, 2006. Nine studies in algebra, geometry and trigonometry from the *Opus Restitutae Mathematicae Analyseos, seu Algebrâ Novâ*, Translated from the Latin and with an introduction by T. Richard Witmer, Reprint of the 1983 original [Kent State Univ. Press, Kent, OH; MR0731262].

[57] Arthur W. Weeks and Jackson B. Adkins. *A Course in Geometry: Plane and Solid.* Ginn and Company, Lexington MA, 1970.