Theories of Action

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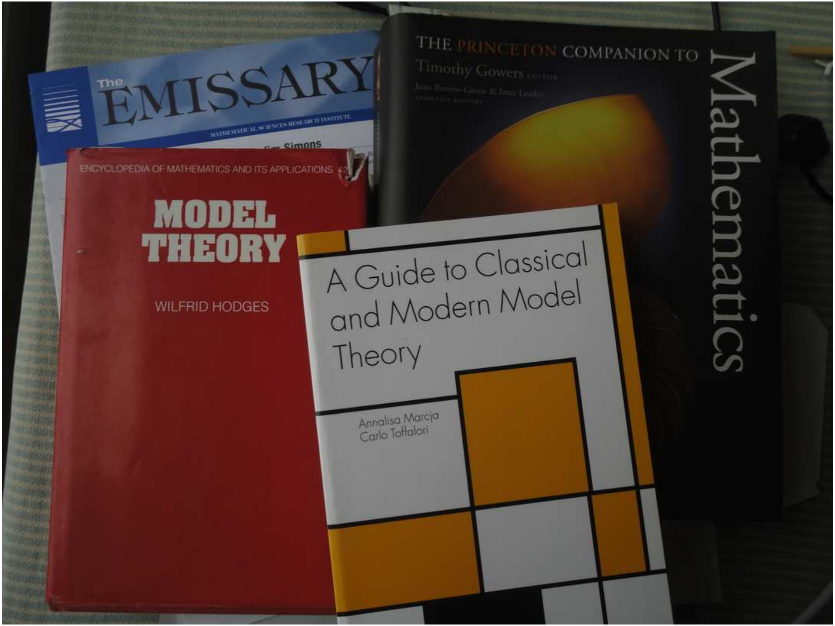

The EMISSARY
MATHEMATICAL SCIENCES RESEARCH INSTITUTE
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
MODEL THEORY
WILFRID HODGES
THE PRINCETON COMPANION TO
Mathematics
Timothy Gowers EDITOR
A Guide to Classical and Modern Model Theory
Annalisa Marcja
Carlo Toffalori

Model theory (I suggest) is mathematics done self-consciously—with awareness of mathematics as an *activity.* Part of the activity is *symbolizing* it.

Marcja & Toffalori (2003) "The study of the truth relation between structures and sentences is just Model Theory, at least according to the feeling in the fifties" [14, p. 8].

Chang & Keisler (1973/1990)

"Universal algebra + logic = model theory" [1, p. 1].

Wilfrid Hodges Model theory is

(1993) "the study of the construction and classification of structures within specified classes of structures" [9, p. ix];

(1997) "algebraic geometry minus fields" [10, p. vii];

(2001/2020) "the study of the interpretation of any language, formal or natural, by means of set-theoretic structures, with Alfred Tarski's truth definition as a paradigm" [11].

Carol Wood (1998) "Geometry with a Twist" [22]:

> In his recent book on model theory, Wilfrid Hodges describes a current view of model theory via the slogan, "model theory is algebraic geometry minus fields." What then is meant by the model theory of fields? . . .
>
> . . . model theory is that branch of logic concerned principally with semantics; its fundamental arena of study is the class of abstract structures and definable sets within these structures, and its most typical language is that of first order logic. The abstraction serves as a lens through which one can understand various parts of mathematics.
>
> It is my impression . . . that non-logicians think of logic in terms of set theory . . .
>
> Recent developments in model theory of fields suggest that the effort required in excising the model theory wherever possible may be greater than learning the model theory involved . . .
>
> . . . .
>
> *Note: I thank Concha Gomez for pointing out the paradox when one combines Hodges' slogan and our program's name!*

Lou van den Dries (1998) "Model theory of fields" [20]:

> In model theory we associate to a structure $\mathfrak{M}$ invariants like $\mathrm{Th}(\mathfrak{M})$ of a logical-combinatorial nature . . .
>
> *Example.* Th($\mathbb{C}$ as ring) is axiomatized by . . .
>
> *Counterexample* (Gödel). Th($\mathbb{Z}$ as ring) cannot be effectively described in any reasonable way. (But $\mathbb{Z}$ as ordered additive group is tame!) . . .
>
> *But:* despite Gödel, mathematical problems, even in apparently "non-tame" subjects like number theory, do get solved, often by ingenious moves into tame territory! Thus the relevance of
>
> model theory ≈ tame mathematics
>
> *Example.* The field $\mathbb{Q}$ of rational numbers is not tame (Julia Robinson), but its completions $\mathbb{R}$, $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, . . . are all tame (Tarski, Ax, Kochen, Eršov). (It is *not known* if the field $\mathbb{F}_p((t))$ is tame.)

≈ is read as "has something to do with" or "is part of" or "is some kind of metamathematics of."

Anand Pillay (2021) "Model theory and groups" [17]:

Model theory studies first order theories T, often complete.

- The study of specific first order theories, such as set theory, or differentially closed fields, can be identified with "applications" of model theory, whereas
- the study of broad classes of first order theories (such as all theories, or stable theories) is what is often considered as "pure" model theory.

There are various invariants of a first order theory T.

- One is the category $\mathbf{Mod}(T)$ of models of T (where the morphisms are elementary embeddings) . . .
- Another invariant is $\mathrm{Def}(T)$, the category of definable sets . . .

David Marker (2008) "Mathematical logic is the study of formal languages that are used to describe mathematical structures and what these can tell us about the structures themselves" [15].

That's from the *Princeton Companion to Mathematics* [6], Part IV, which covers the following "Branches of Mathematics": (1) Algebraic Numbers; (2) Analytic Number Theory; (3) Computational Number Theory; (4) Algebraic Geometry; (5) Arithmetic Geometry; (6) Algebraic Topology; (7) Differential Topology; (8) Moduli Spaces; (9) Representation Theory; (10) Geometric and Combinatorial Group Theory; (11) Harmonic Analysis; (12) Partial Differential Equations; (13) General Relativity and the Einstein Equations; (14) Dynamics; (15) Operator Algebras; (16) Mirror Symmetry; (17) Vertex Operator Algebras; (18) Enumerative and Algebraic Combinatorics; (19) Extremal and Probabilistic Combinatorics; (20) Computational Complexity; (21) Numerical Analysis; (22) Set Theory; (23) Logic and Model Theory; (24) Stochastic Processes; (25) Probabilistic Models of Critical Phenomena; (26) High-Dimensional Geometry and Its Probabilistic Analogues.

We pass to model theory in action, *of* action. Let

- $\boxed{F}$ be a set of "functions,"
- $\boxed{P}$ be a set of "points,"
- $\boxed{\Phi}$ be a function from $F \times P$ to P.

A *structure* (F, P, Φ) is a **group action** if, each $\Phi(\alpha, b)$ written as $\alpha\, b$,

1) all functions have inverses: $\forall\xi\ \exists\eta\ \forall z\ (\xi\,\eta\,z = z \wedge \eta\,\xi\,z = z)$;

2) any two functions have a composite: $\forall\xi\ \forall\eta\ \exists\zeta\ \forall x\ \xi\,\eta\,x = \zeta\,x$;

3) there is an identity: $\exists\xi\ \forall y\ \xi\,y = y$.

These are *sentences* of the *first-order logic* in the *signature* $\{F, P, \Phi\}$. They axiomatize a *theory,* $\boxed{\mathsf{GA}}$, of group actions. Axiomatizing a subtheory $\boxed{\mathsf{A}}$, of **actions,** are the conditions

4) all functions are surjective: $\forall\xi\ \forall y\ \exists z\ \xi\,z = y$;

5) all functions are injective: $\forall\xi\ \forall z\ \forall z'\ (\xi\,z = \xi\,z' \Rightarrow z = z')$.

A **structure** assigns, to each symbol in its signature, an **interpretation.**

The structure being $\mathfrak{A}$; the symbol, S; the interpretation can be $\boxed{S^{\mathfrak{A}}}$.

The signature being $\{F, P, \Phi\}$,

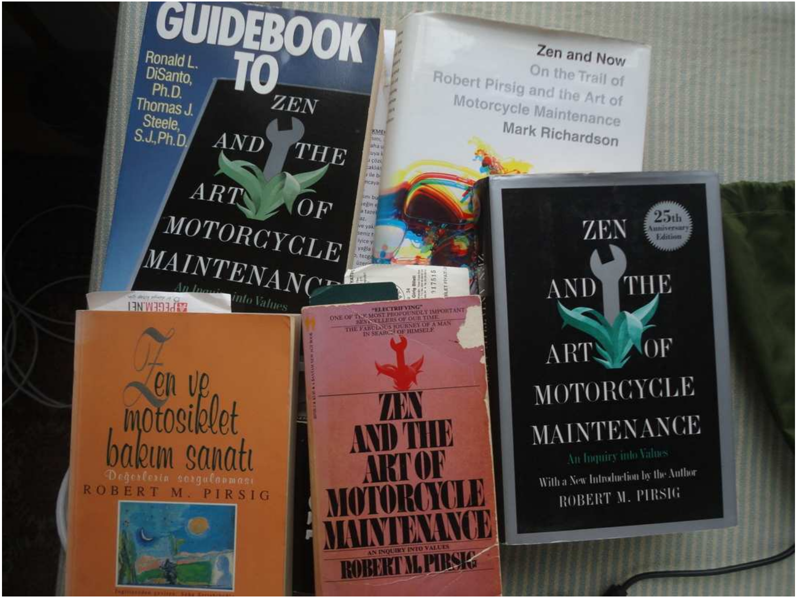
- $F^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$ are sets;
- $\Phi^{\mathfrak{A}} \colon F^{\mathfrak{A}} \times P^{\mathfrak{A}} \to P^{\mathfrak{A}}$.

We can then write $\mathfrak{A}$ as $(F^{\mathfrak{A}}, P^{\mathfrak{A}}, \Phi^{\mathfrak{A}})$ or just (F, P, Φ).

F and P are **sorts.** With a unique sort, $\mathfrak{A}$ could be $(A, {S_0}^{\mathfrak{A}}, {S_1}^{\mathfrak{A}}, \dots)$.

Given a signature, we recursively define

- *formulas,* using, additionally, $\vee$, $\neg$, $\exists$, and variables for each sort—symbols $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ are used in abbreviations;
- the **truth value** (true or false) of formulas in a given structure under a given assignment of values, each from the right sort, to the *free variables*; this needs the derivation of a formula to be unique.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Calling himself Phaedrus, Robert Pirsig (1974) describes political acts that affected the activity of his university in Montana [18, ch. 13]:

- Professors were told that all public statements must be cleared through the college public-relations office before they could be made.
- . . . the legislature had passed a law fining the college $8000 for every student who failed . . .
- The newly elected governor was trying to fire the college president for both personal and political reasons . . .
- . . . funds to the college were being cut. The college president had passed on an unusually large part of the cut to the English department, of which Phaedrus was a member, and whose members had been quite vocal on issues of academic freedom.

Phaedrus . . . was exchanging letters with the Northwest Regional Accrediting Association to see if they could help prevent these violations of accreditation requirements . . .

A student opposed Pirsig, asking

> if . . . he was trying to prevent them from getting an education.

Another student

> said angrily that the legislature would prevent the school from losing its accreditation . . . they would post police to prevent it.

Pirsig lectured next day in response:

> The real University is a state of mind. It is that great heritage of rational thought that has been brought down to us through the centuries . . .
>
> In addition to this state of mind, "reason," there's a legal entity which is unfortunately called by the same name but which is quite another thing . . .
>
> Confusion continually occurs in people who fail to see this difference . . . They see professors as employees of the second university who should abandon reason when told to and take orders with no backtalk, the same way employees do in other corporations.

Back to model theory. A formula with no free variables is a **sentence.** The structures in which a set Γ of sentences are true are the **models** of Γ; let them compose $\boxed{\mathbf{Mod}(\Gamma)}$. This reverses inclusions:

$$\Gamma \subseteq \Delta \quad \text{implies} \quad \mathbf{Mod}(\Gamma) \supseteq \mathbf{Mod}(\Delta).$$

Some sentences σ can be formally **proved** from Γ; write then $\boxed{\Gamma \vdash \sigma}$. The set $\{\sigma \colon \Gamma \vdash \sigma\}$ is the **theory axiomatized** by Γ. Then

$$\mathbf{Mod}(\{\sigma \colon \Gamma \vdash \sigma\}) = \mathbf{Mod}(\Gamma).$$

A theory containing $\neg\sigma$ for every σ not in it is **complete.** For example, writing

- for "σ is true in $\mathfrak{A}$," $\boxed{\mathfrak{A} \models \sigma}$,
- for $\{\sigma \colon \mathfrak{A} \models \sigma\}$, $\boxed{\mathrm{Th}(\mathfrak{A})}$ ("the theory of $\mathfrak{A}$"),

we have that $\mathrm{Th}(\mathfrak{A})$ is always a complete theory.

Of any theory T (such as our GA or A), we may ask:

- What are its completions U?
- For which U is $\mathbf{Mod}(U)$ the most tractable?

If T is field theory, then the "best" U are the theories ACF_p of algebraically fields of characteristic p, where (p) is a prime ideal of $\mathbb{Z}$.

The $\cong$-class of a model K of ACF_p is determined by tr-deg(K).

Thus $\mathbf{Mod}(\mathsf{ACF}_p)/\cong$ is a chain or tower.

Such a neat result is rare.

We give some details on formulas and their interpretations in structures.

1. *Terms* (such as polynomials in field theory) encode operations.
2. Terms are combined into *atomic formulas* (such as polynomial equations), which have solution sets.
3. On such sets, the results of operations such as $\cap$, $\cup$, $\smallsetminus$, and coordinate projection are encoded in arbitrary *first-order formulas.*

In our logic of $\{F, P, \Phi\}$,

- An F**-term** is one of the variables $\boxed{\xi,\ \eta,\ \zeta,\ \dots}$
- The P**-terms** are given recursively:
  - each of the variables $\boxed{x,\ y,\ z,\ \dots}$ is a P-term;
  - if ϑ is an F-term; t, a P-term; then $\boxed{\vartheta\, t}$ is a P-term.

Thus $\zeta\,\eta\,x$ is a P-term—but $\zeta\,\eta$ is not an F-term. An equation $\boxed{t_0 = t_1}$ of F-terms or P-terms is an **atomic formula.** This has, in every $\mathfrak{A}$ of the signature, a solution set, $\boxed{(t_0 = t_1)^{\mathfrak{A}}}$, comprising certain tuples with an entry for each variable. Operations on solution sets are encoded in **first-order formulas,** recursively:

$$(\varphi \lor \psi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}} \cup \psi^{\mathfrak{A}}, \qquad (\neg\varphi)^{\mathfrak{A}} = \left(\varphi^{\mathfrak{A}}\right)^{\mathrm{c}}, \qquad (\exists \mathfrak{x}\ \varphi)^{\mathfrak{A}} = \pi_{\mathfrak{x}}\left[\varphi^{\mathfrak{A}}\right],$$

$\pi_{\mathfrak{x}}$ being the projection deleting the $\mathfrak{x}$-entry ($\mathfrak{x}$ standing for a variable). For sentences σ (such as $\exists \mathfrak{x}\ \varphi$), $\sigma^{\mathfrak{A}}$ is $\varnothing$ or $\{\varnothing\}$, and

$$\mathfrak{A} \vDash \sigma \quad (\text{iff} \quad \varphi^{\mathfrak{A}} \neq \varnothing) \quad \text{iff} \quad \sigma^{\mathfrak{A}} \neq \varnothing.$$

A class $\mathbf{Mod}(\Gamma)$ is called **elementary;** it is $\bigcap\bigl\{\mathbf{Mod}(\{\sigma\})\colon \sigma \in \Gamma\bigr\}$. For every class $\boldsymbol{K}$ of structures, we define

$$\bigcap\{\mathrm{Th}(\mathfrak{A})\colon \mathfrak{A} \in \boldsymbol{K}\} = \mathrm{Th}(\boldsymbol{K}).$$

This too reverses inclusion: $\boldsymbol{K} \subseteq \boldsymbol{L}$ implies $\mathrm{Th}(\boldsymbol{K}) \supseteq \mathrm{Th}(\boldsymbol{L})$. Also

$$\{\sigma\colon \Gamma \vdash \sigma\} \subseteq \mathrm{Th}(\mathbf{Mod}(\Gamma)).$$

Gödel Completeness (1930) The reverse inclusion holds [4], so there is a Galois correspondence between theories and elementary classes.

Compactness Proofs being finite, if Γ has no models, some finite subset must not. Thus, since

$$\mathbf{Mod}(\{\sigma\}) \cup \mathbf{Mod}(\{\tau\}) = \mathbf{Mod}(\{\sigma \vee \tau\}),$$

these are the basic closed classes in a compact topology on $\mathbf{Mod}(\varnothing)$.

Presburger (1929) $\mathrm{Th}(\mathbb{N}, +)$ can be axiomatized [19].

Gödel Incompleteness (1931) $\mathrm{Th}(\mathbb{N}, +, \times)$ cannot be axiomatized [5].

Ultraproducts (Łoś 1955) Each K_i being a field, let $\prod(K_i\colon i\in I)$ have prime ideal $\mathfrak{P}$. Now define

$$\Bigl\{\{i\in I\colon a_i\neq 0\}\colon (a_i\colon i\in I)\in\prod(K_i\colon i\in I)\Bigr\}=\mathfrak{p};$$

its elements are "small" subsets of I. Then for all σ with parameters, $(a_i\colon i\in I)^{K_j}$ being a_j,

$$\Bigl(\prod(K_i\colon i\in I)\Bigr)\Big/\mathfrak{P}\vDash\sigma\quad\text{iff}\quad\{i\in I\colon K_i\vDash\neg\sigma\}\in\mathfrak{p}.$$

Proof. With $\{i\in I\colon K_i\vDash\neg\sigma\}=\|\sigma\|$, by induction:

1. It's true when σ is atomic, that is, a polynomial equation.
2. $\|\neg\sigma\|=I\smallsetminus\|\sigma\|$, and $X\in\mathfrak{p}$ iff $I\smallsetminus X\notin\mathfrak{p}$.
3. $\|\sigma\lor\tau\|=\|\sigma\|\cap\|\tau\|$, and $X\cap Y\in\mathfrak{p}$ iff one of X and Y is in $\mathfrak{p}$.
4. $\|\exists x\ \varphi(x)\|=\|\varphi(a)\|$, where $K_i\vDash\varphi(a_i)$ iff $K_i\vDash\exists x\ \varphi(x)$. □

The proof of Łoś's Theorem works for structures in any signature and yields the following

Proof of Compactness. Let the finite subsets of Γ compose I, and for each i in I, suppose

$$\mathfrak{A}_i \vDash i.$$

Writing, for each j in I,

$$\{i \in I \colon j \nsubseteq i\} = [j],$$

we have

$$\{i \in I \colon \mathfrak{A}_i \notin \mathbf{Mod}(j)\} \subseteq [j], \qquad [j] \cup [k] = [j \cup k].$$

so the $[j]$ generate a proper ideal of $\mathscr{P}(I)$, included in a prime ideal $\mathfrak{p}$, and when we let

$$(a_i \colon i \in I) \sim (b_i \colon i \in I) \quad \text{iff} \quad \{i \in I \colon a_i \neq b_i\} \in \mathfrak{p},$$

by Łoś we have

$$\Big(\prod(\mathfrak{A}_i \colon i \in I)\Big)\Big/{\sim} \vDash \Gamma. \qquad \square$$

Our formulas are **first-order** because

- the variables stand for individuals, not sets of them;
- only finitely many atomic formulas occur in any formula.

Non-standard analysis uses that

- $\mathrm{Th}(\mathbb{R})$ is $\mathrm{Th}(\mathbb{R}^{\mathbb{N}}/\mathfrak{P})$ (the first-order theories are the same);
- $\mathbb{R}$ is complete as an ordered field, but not $\mathbb{R}^{\mathbb{N}}/\mathfrak{P}$, because it has nonzero infinitesimals (if $\mathfrak{P}$ is non-principal).

Because the spaces of their models are not compact, the following conditions are not first-order:

- Being inductive: $\forall X\ \big(1 \in X \land \forall y\ (y \in X \Rightarrow y' \in X) \Rightarrow \forall y\ y \in X\big)$.
- Being finite: $\bigvee_{n\in\mathbb{N}} \big(\exists x_0 \cdots \exists x_{n-1}\ \forall y\ \bigvee_{i<n} y = x_i\big)$.
- Being a simple group (since there are arbitrarily large finite abelian simple groups, namely the $\mathbb{Z}/p\mathbb{Z}$, but no infinite ones).

The algebraically closed fields are the **existentially closed** models of the theory of fields; for, they have a solution to each finite system of equations and inequations that is soluble in some larger field.

The existentially closed models of A are those of A^*, axiomatized by the following sentences (the more formal versions are on the next slide):

1. The axioms of A.

2. There are at least two distinct points.

3. For every *proper* number n (so $n \geqslant 2$),

   a) for every set of n points, there are $n!$ functions serving as the full symmetry group;

   b) every set of $n!$ functions serves (in every way) as the full symmetry group of some set of n points.

Because also the axioms of A are $\forall\exists$, which means the union of a chain of models is a model, therefore A^* is the **model-companion** of A.

Formally, the axioms of A^* are:

$$\left.\begin{array}{c}\forall\xi\ \forall y\ \exists z\ \xi\, z = y,\\ \forall\xi\ \forall y\ \forall z\ (\xi\, y = \xi\, z \Rightarrow y = z),\end{array}\right\}\ \text{axioms of } \mathsf{A}$$

$$\exists x\ \exists y\ x \neq y,$$

and, n being at least 2, $\boxed{\varphi_n(\boldsymbol{\xi}, \boldsymbol{x}) \text{ being } \bigwedge_{\sigma\in \mathrm{S}_n} \bigwedge_{i<n} \xi_\sigma\, x_i = x_{\sigma(i)}}$,

$$\forall \boldsymbol{x}\ \exists \boldsymbol{\xi}\ \left(\bigwedge_{i<j<n} x_i \neq x_j \Rightarrow \varphi_n(\boldsymbol{\xi}, \boldsymbol{x})\right),$$

$$\forall \boldsymbol{\xi}\ \exists \boldsymbol{x}\ \left(\bigwedge_{\substack{\sigma\neq\tau\\ \{\sigma,\tau\}\subseteq \mathrm{S}_n}} \xi_\sigma \neq \xi_\tau \Rightarrow \varphi_n(\boldsymbol{\xi}, \boldsymbol{x}) \wedge x_0 \neq x_1\right).$$

A^* is the **model-companion** of A *and* of GA, because:

- Every model of A embeds in a model of A^*.
- Every model (F, P) of A^* embeds in the model $\big(\mathrm{Sym}(P), P\big)$ of GA.
- Every model of A^* is existentially closed—A^* is **model-complete**—because in models of A^*, for example,

$$\exists\zeta\ (\zeta\, x_0 = y_0 \wedge \zeta\, x_1 = y_1) \Leftrightarrow (x_0 = x_1 \Leftrightarrow y_0 = y_1),$$
$$\exists z\ (\xi_0\, z = y_0 \wedge \xi_1\, z = y_1) \Leftrightarrow {\xi_0}^{-1}\, y_0 = {\xi_1}^{-1}\, y_1,$$

  and in general there is full elimination of quantifiers in the theory in (F, P, Φ, Ψ) whose axioms are those of A^*, along with

$$\forall\xi\ \forall y\ \forall z\ (\xi\, y = z \Leftrightarrow y = \xi^{-1}\, z),$$

  $\alpha^{-1}\, b$ standing for $\Psi(\alpha, b)$, where $\Psi\colon F \times P \to P$.

Moreover $\boxed{\mathsf{A}^* \text{ is complete}}$, because there are no quantifier-free sentences.

The quantifier elimination yields that A^* is the *model-completion* of A.

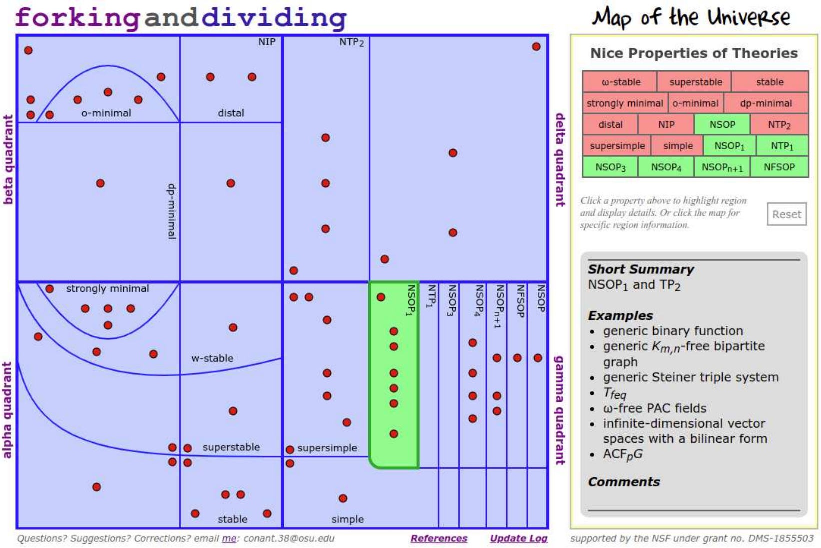
- ACF, though not complete, is the model-completion of Th(fields);
- $\mathrm{Th}(\mathbb{C}/\mathbb{Q}^{\mathrm{alg}})$, though complete, is not model-complete, because the existentially closed models L/K have $\text{tr-deg}(L/K) = 1$.

$$\text{When } \left\{ \begin{aligned} T_\forall &= \mathrm{Th}(\text{substructures of models of } T), \\ \mathrm{diag}(\mathfrak{A}) &= \mathrm{Th}(\text{structures in which } \mathfrak{A} \text{ embeds}) \end{aligned} \right\}, \text{ then}$$

- $T_\forall$ is axiomatized by the universal sentences in T;
- $\mathrm{diag}(\mathfrak{A})$ is axiomatized by the atomic and negated atomic sentences (with parameters) true in $\mathfrak{A}$.

Say $U_\forall = T_\forall$. If $T \cup \mathrm{diag}(\mathfrak{A})$ axiomatizes a complete theory whenever

- $\mathfrak{A} \models T$, then T is **model-complete** and the **model-companion** of U;
- $\mathfrak{A} \models U$, and $U \subseteq T$, then T is the **model-completion** of U;
- $\mathfrak{A} \models T_\forall$, then T admits quantifier elimination.



Gabriel Conant, `https://forkinganddividing.com/`

Letting ω be $\{0\} \cup \mathbb{N}$, letting $(i, j) \mapsto a^i_j$ embed $\omega \times \omega$ in ω (for example, $a^i_j = (i + j)^2 + i$), form the array

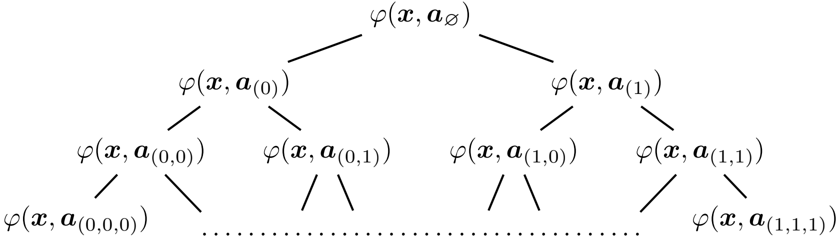
$$
\begin{array}{ccccc}
\xi\,0 = a^0_0 & \xi\,0 = a^0_1 & \xi\,0 = a^0_2 & \xi\,0 = a^0_3 & \dots \\
\xi\,1 = a^1_0 & \xi\,1 = a^1_1 & \xi\,1 = a^1_2 & \xi\,1 = a^1_3 & \dots \\
\xi\,2 = a^2_0 & \xi\,2 = a^2_1 & \xi\,2 = a^2_2 & \xi\,2 = a^2_3 & \dots \\
\multicolumn{5}{c}{\dotfill}
\end{array}
$$

of formulas of $\{F, P, \Phi\}$ with P-parameters from ω. In a model of A^* with $\omega \subseteq P$, a system of

- two equations from the same row of the array is insoluble;
- any number of equations from different rows is soluble.

Therefore A^* has TP_2 ("tree property of the second kind"). That's bad. At least A^* is also NSOP_1 ("not the strong order property of the first kind"):

A complete theory has SOP_1 if there is a tree



for some formula $\varphi(\boldsymbol{x}, \boldsymbol{y})$ and binary tree $(\boldsymbol{a}_\sigma \colon {}^{<\omega}2)$ of parameters from a model, such that a system of

- two formulas from different branches is insoluble (at least if the first difference in their indices is in the last entry of one),
- any number of formulas from a branch is soluble.

A^* hasn't got SOP_1. How does one show that?

Chernikov and Ramsey [2] show that a theory's *not* having SOP_1 is implied by *having* a ternary relation $\downarrow$ with certain properties on sets of parameters from a model.

- In an algebraically closed field, the relation is algebraic independence. If A, B, and C are subsets, the last two generating K and L,

$$A \underset{B}{\downarrow} C \quad \text{iff} \quad \bigwedge_{X \in \mathscr{P}_\omega(A)} \text{tr-deg}(L(X)/L) = \text{tr-deg}(K(X)/K).$$

- In a separably closed field K of characteristic 2 satisfying

$$\forall x\ \exists x_0\ \exists x_1\ x = {x_0}^2 + {x_1}^2 b,$$

each a in K determines a binary tree $(a_\sigma \colon \sigma \in {}^{<\omega}2)$, where

$$a_{(\sigma_0,\dots,\sigma_n)} = {a_{(\sigma_0,\dots,\sigma_{n-1},0)}}^2 + {a_{(\sigma_0,\dots,\sigma_{n-1},1)}}^2 b,$$

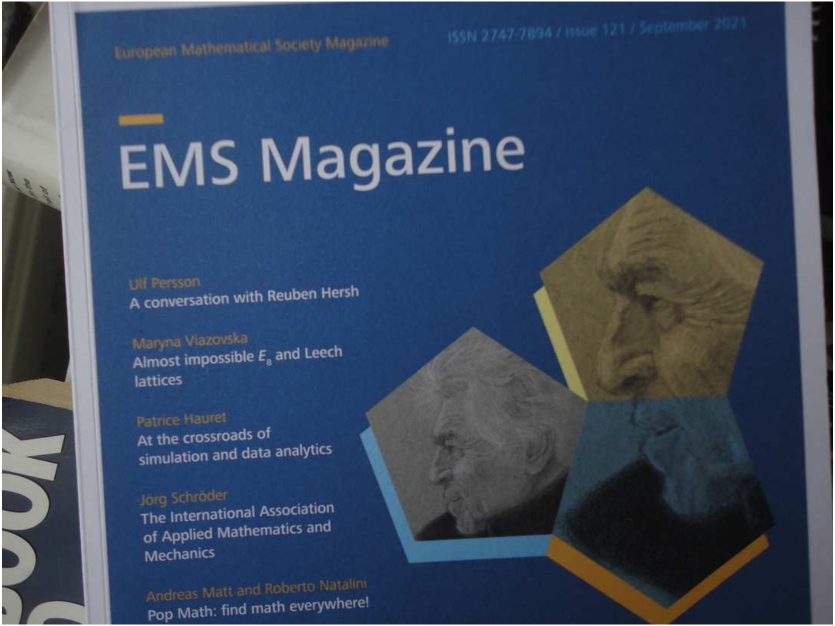
and then we replace (X) above with $(\{x_\sigma \colon x \in X \wedge \sigma \in {}^{<\omega}2\})$.

- In a model (F, P, Φ) of A^*, $(B, R) \mathop{\smile\!\!\!\!\!\!|}_{(A,Q)} (C, S)$ means (B, R) and (C, S) have nothing in common that is not already in (A, Q). In detail:

  1. A subset A of F generates, of $\mathrm{Sym}(P)$, the subgroup $\langle A\rangle$.
  2. An element p of P has, under $\langle A\rangle$, the orbit $\langle p\rangle_A$.
  3. If $Q \subseteq P$, then $\bigcup\{\langle x\rangle_A : x \in Q\} = \langle Q\rangle_G$.
  4. If $A \subseteq B \cap C$ in F and $Q \subseteq R \cap S$ in P, we define

$$(B, R) \mathop{\smile\!\!\!\!\!\!|}_{(A,Q)} (C, S) \quad \text{iff} \quad \begin{cases} B \cap C = A, \\ \langle R\rangle_B \cap \langle S\rangle_C = \langle Q\rangle_A. \end{cases}$$

The properties of $\mathop{\smile\!\!\!\!\!\!|}$ enjoyed by the examples and sufficient for NSOP_1 are called (1) strong finite character, (2) existence over models, (3) monotonicity, (4) symmetry, (5) independent amalgamation.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Ulf Persson (2021) “A conversation with Reuben Hersh” [16]:

Ulf Persson . . . What is true in mathematics is not up to our discretion, certainly not as individuals.

Reuben Hersh But in practice truth is agreed on by a process of social confirmation . . .

UP Sure . . .

RH So you agree, even when it comes to truth in mathematics it is a matter of social convention.

UP But the remarkable thing is that this convention is so consensual . . . I think that there is something beyond the practice of mathematics, beyond the human fallible way of doing mathematics . . . if there is a counter-example to a previously authorized theorem, that will surely trump . . .

RH Absolutely right. Nevertheless . . . Mathematical Platonism is a . . . fallacy. It arises from the unfounded idea that there must be something to mathematics beyond the practice of mathematics.

Is the following Platonism?

G. H. Hardy (1928) "Mathematical Proof," *Mind* [8]:

(1) . . . no philosophy can possibly be sympathetic to a mathematician which does not admit . . . the immutable and unconditional validity of mathematical truth . . .

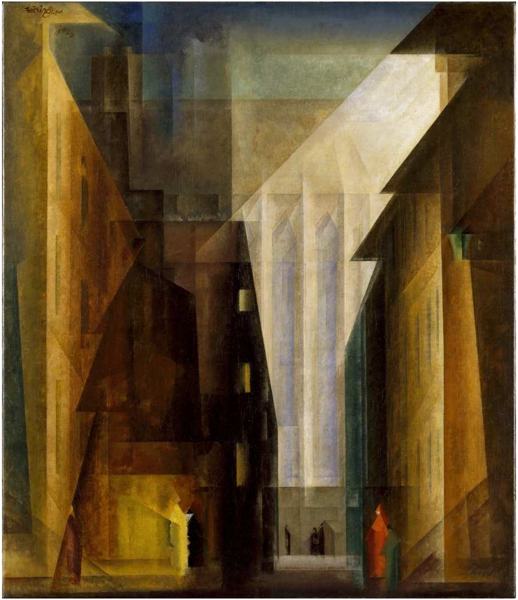
(2) When we know a mathematical theorem, there is something, some object, which we know . . .

(3) . . . the vast majority of mathematicians will rebel against the doctrine . . . that it is only the so-called 'finite' theorems of mathematics which possess a real significance. That 'the finite cannot understand the infinite' should surely be a theological and not a mathematical war-cry.

Øystein Linnebo (2009/2018) "Platonism in the Philosophy of Mathematics," *The Stanford Encyclopedia of Philosophy* [12]:

> Mathematical platonism can be defined as the conjunction of the following three theses:
>
> **Existence.**
>
> There are mathematical objects.
>
> **Abstractness.**
>
> Mathematical objects are abstract.
>
> **Independence.**
>
> Mathematical objects are independent of intelligent agents and their language, thought, and practices.
>
> . . . **Abstractness** says that every mathematical object is abstract, where an object is said to be abstract just in case it is non-spatiotemporal and (therefore) causally inefficacious.

Does truth not cause us to strive for it?



Lyonel Feininger
Church of the Minorites
1926
Walker Art Center
Minneapolis

Robert Pirsig (1974) on university as "Church of Reason" [18, ch. 13]:

> The primary goal of the Church of Reason . . . is always Socrates' old goal of truth, in its ever-changing forms, as it's revealed by the process of rationality. Everything else is subordinate to that. Normally this goal is in no conflict with the location goal of improving the citizenry, but on occasion some conflict arises, as in the case of Socrates himself. It arises when trustees and legislators who've contributed large amounts of time and money to the location take points of view in opposition to the professors' lectures or public statements. They can then lean on the administration by threatening to cut off funds if the professors don't say what they want to hear. That happens too.
>
> True churchmen in such situations must act as though they had never heard these threats. Their primary goal never is to serve the community ahead of everything else. Their primary goal is to serve, through reason, the goal of truth.
>
> That was what he meant by the Church of Reason.

There are "Replication crises" in science: "A massive 8-year effort finds that much cancer research can't be replicated" (Tara Haelle, *Science News,* 2021 [7]).

Richard Feynman (1974) "Cargo Cult Science: Some remarks on science, pseudoscience, and learning how to not fool yourself," [3]:

> In order to compare his heavy hydrogen results to what might happen to light hydrogen he had to use data from someone else's experiment on light hydrogen, which was done on different apparatus . . . he couldn't get time on the program . . . to do the experiment with light hydrogen on this apparatus because there wouldn't be any new result. And so the men in charge of programs at [National Accelerator Laboratory] are so anxious for new results, in order to get more money to keep the thing going for public relations purposes, they are destroying—possibly—the value of the experiments themselves, which is the whole purpose of the thing. It is often hard for the experimenters there to complete their work as their scientific integrity demands.

There is a corresponding danger for mathematics.

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