A Construction of the Regular Heptakaidekagon

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Contents

1 Introduction

This is about

- how to construct a *regular heptakaidecagon* and
- why the construction works.

We shall justify everything in terms of the *Elements* of Euclid [4, 5, 6, 7].

A **heptakaidecagon** is a polygon with seventeen sides; a **regular** polygon has all sides equal to one another and all angles equal to one another. We shall be inscribing a regular heptakaidecagon in a given circle.

Such an inscription, or at least its existence, was first announced in 1796 [1] by Carl Friedrich Gauss, who had been born in 1777 and who included his result in the seventh and last section, called "Equations Defining Sections of a Circle," of his *Disquisitiones Arithmeticae* [8, pages 407–36], published in 1801.

The treatment below will be based on Gauss's and on that of Hardy and Wright in *An Introduction to the Theory of Numbers* [9, pages 58–62] (first edition, 1938; fifth edition, 1979). The latter treatment is nearly copied (with attribution) from Richmond, "A Construction for a Regular Polygon of Seventeen Sides" [12] (1893).

Since all of Gauss's work can be expressed in Euclidean terms, we may ask why Euclid or Archimedes did not actually do it (as far as we know).

Euclid constructed *some* regular polygons. Indeed, the first proposition of the first book of the *Elements* is to construct an equilateral triangle or "trigon." As we learn in the fifth proposition, the equilateral triangle is also equiangular and is therefore regular.

The regular quadrilateral—or quadrangle or "tetragon"—is a square, constructed in Proposition 46.

In Book IV of the *Elements,* Euclid constructs regular polygons with five, six, and fifteen sides.

The construction of the regular pentagon may be more difficult than any earlier proposition.

The construction of the regular pentekaidekagon works because

- $15 = 3 \cdot 5$,
- 3 and 5 are prime and therefore prime to one another.

Euclid treats numbers, prime and otherwise, in Books VII–IX of the *Elements.* He presumably recognizes that, if k is prime to n, and the regular k-gon and n-gon are constructible, then so is the regular kn-gon. This is true because, by what we call the Euclidean Algorithm, developed in the first propositions of Book VII, we can find counting numbers that solve one of the equations

$$kx = ny + 1, \qquad ny = kx + 1.$$

Correspondingly,

$$\frac{x}{n} - \frac{y}{k} = \frac{1}{kn}, \qquad \frac{y}{k} - \frac{x}{n} = \frac{1}{kn},$$

and thus, from the nth and kth parts of the circle, we can obtain the knth part.

If the regular n-gon is constructible, then so is the regular $2n$-gon: Euclid will use this in establishing Proposition 2 of Book XII, that circles are to one another in the ratio of the squares on their diameters.

Perhaps that proposition is the deepest of the *Elements.* It is summarized today by the equation

$$A = \pi r^2$$

for the area of a circle in terms of the radius. If people are afraid of an equation like that, it may be for good reason, because it hides all of the work needed for Euclid's proof. That proof uses the principle on which today's calculus is based: that if two magnitudes have a ratio, in the sense that some multiple of the less exceeds the greater, then the excess of the greater over the less also has a ratio to the greater (hence to the less as well).

Euclid uses this principle already in Proposition 8 of Book V, the book that lays out the abstract theory of ratio.

Probably Euclid and other Ancients tried to construct a regular heptagon. They failed. I do not know whether they then tried to prove that the construction was impossible.

Euclid does prove that *some* things are impossible. He does it in each case of what we call a proof by contradiction. The first example is in Proposition I.6, that a triangle with equal base angles is isosceles, because its *not* being isosceles is impossible.

By what we have said, Euclid could construct a regular polygon whose number of sides is any of

$$3, \quad 4, \quad 5, \quad 6, \quad 8, \quad 10, \quad 12, \quad 15, \quad 16.$$

The regular heptagon (for example) cannot be constructed, because, briefly,

- 7 is prime;
- $7 - 1 = 6$,
- $6 = 2 \cdot 3$.

The problem is the factor 3. Constructing the regular heptagon would require us to take *cube* roots, and we cannot do that with ruler and compass. We can take *square* roots, and this, briefly, is why we can construct the regular heptakaidecagon:

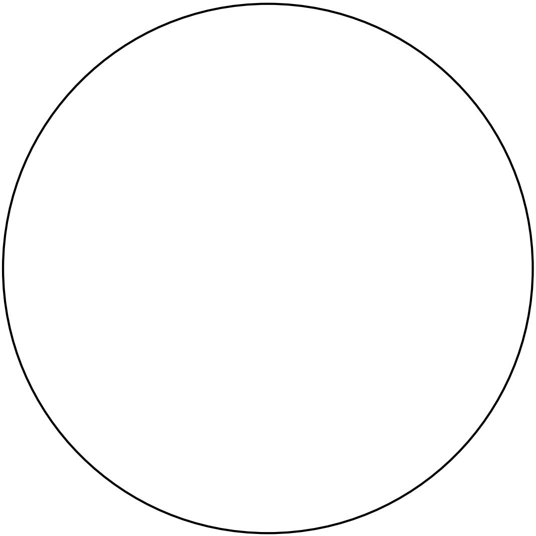
- 17 is prime,
- $17 - 1 = 16$,
- 16 is a power of 2 (namely 2^4).

Now we work out the details.

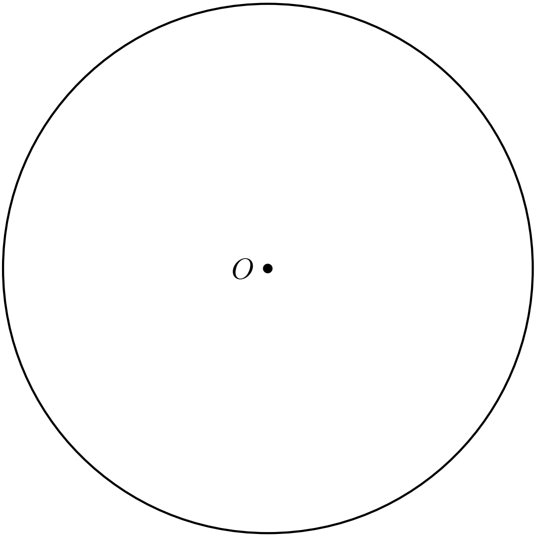
2 Construction

First we just lay out the steps of the construction of the regular heptakaidecagon, giving no reason why they do have that result (this comes in Chapter 3).

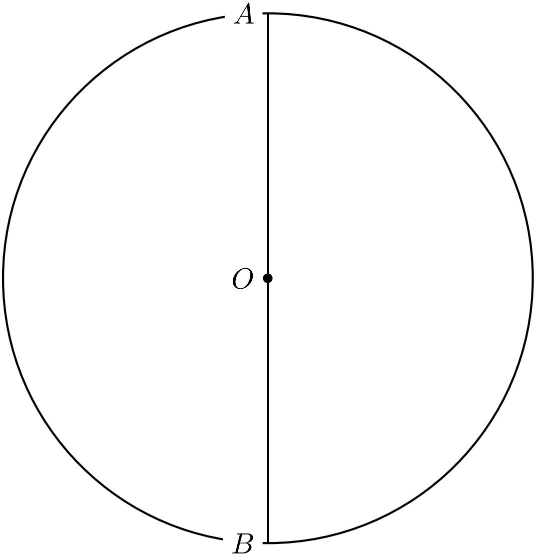
We are given a circle.



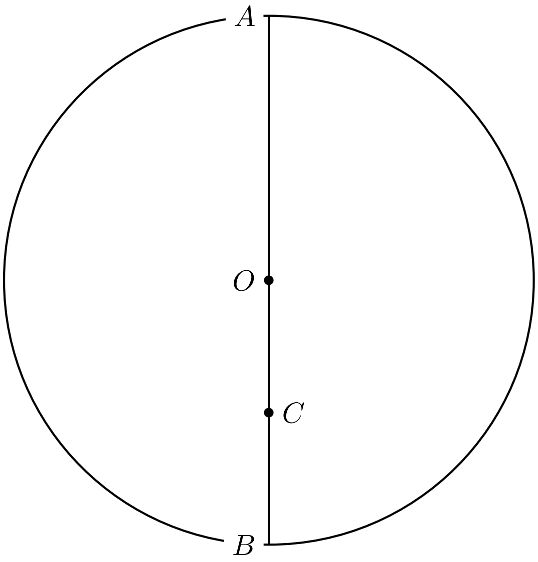
By Proposition III.1 of the *Elements*, we can construct the center of the circle, which we label as O.



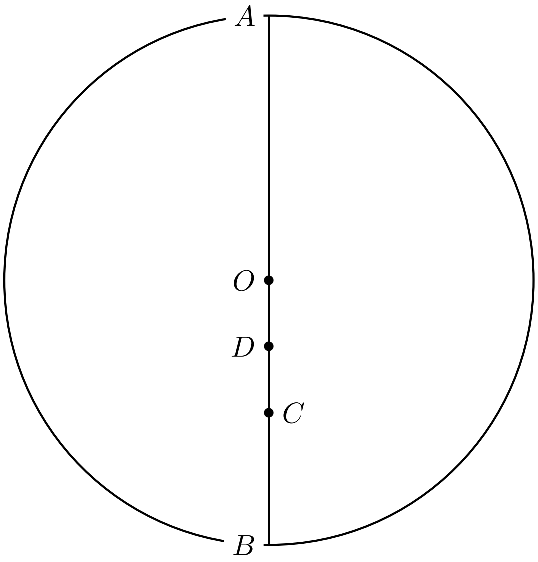
We label a diameter as AB.



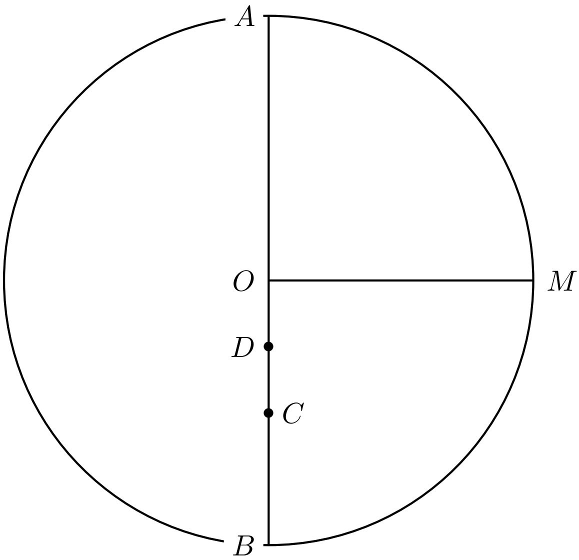
By Proposition I.10 of the *Elements,* we bisect OB, letting the midpoint be C.



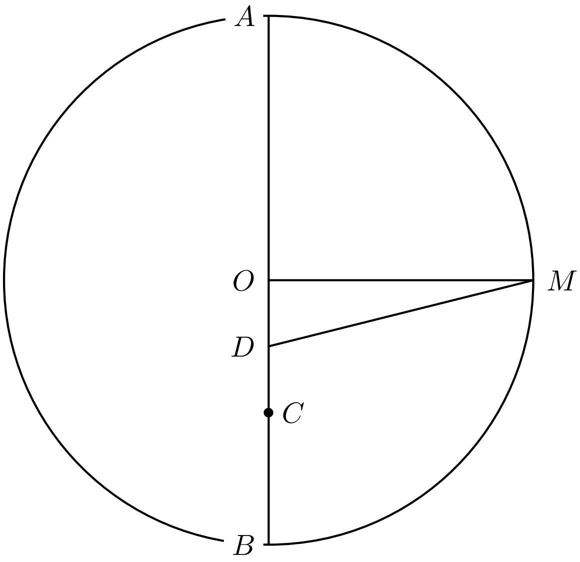
We likewise let the midpoint of OC be D.



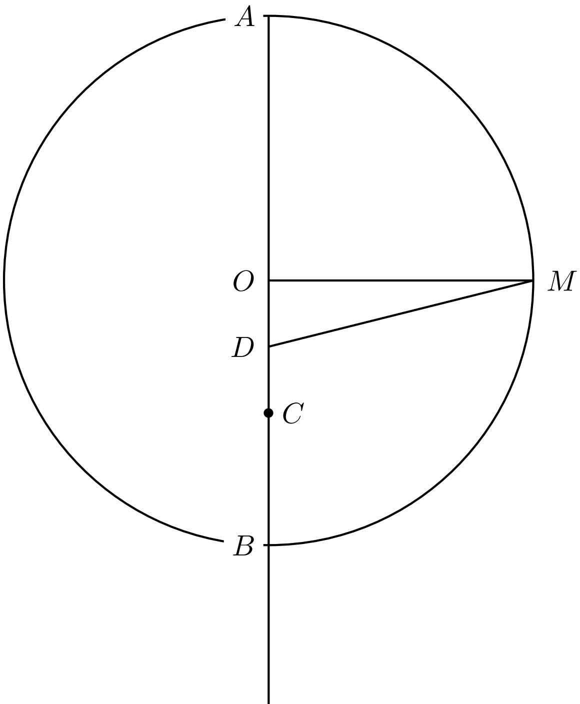
By Proposition I.11 of the *Elements*, we construct the perpendicular to AB at O, meeting the circle at M.



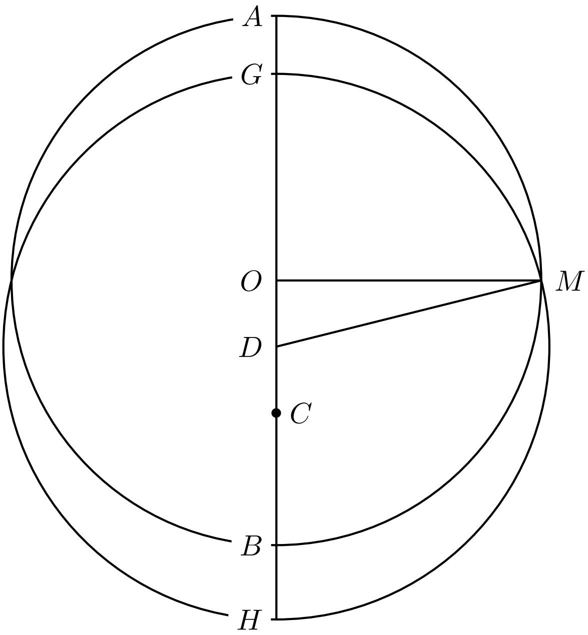
We join DM.



We extend AB.

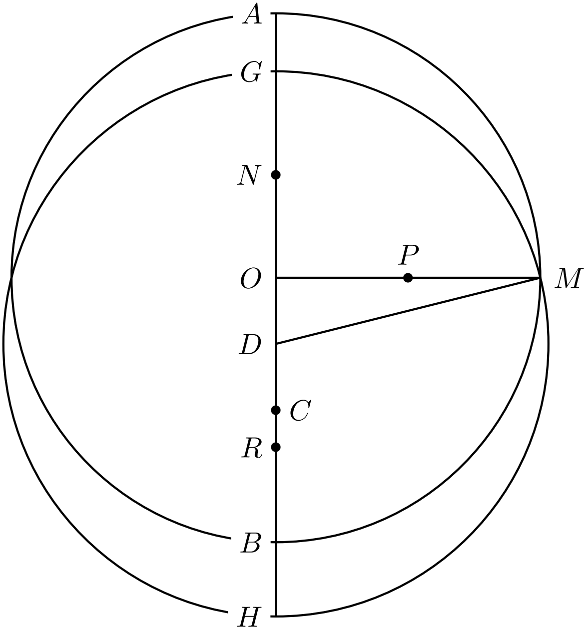


We construct a circle with center O and distance DM, meeting AB at G and the extension of AB at H.

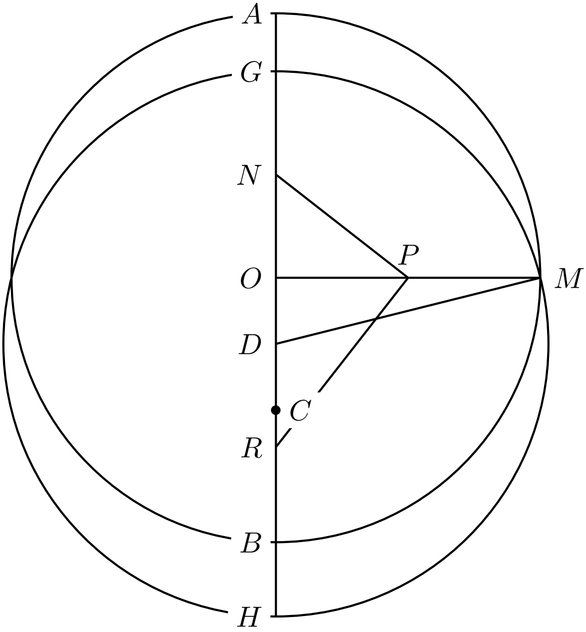


We let the midpoint of

- OG be N,
- OM be P,
- OH be R.

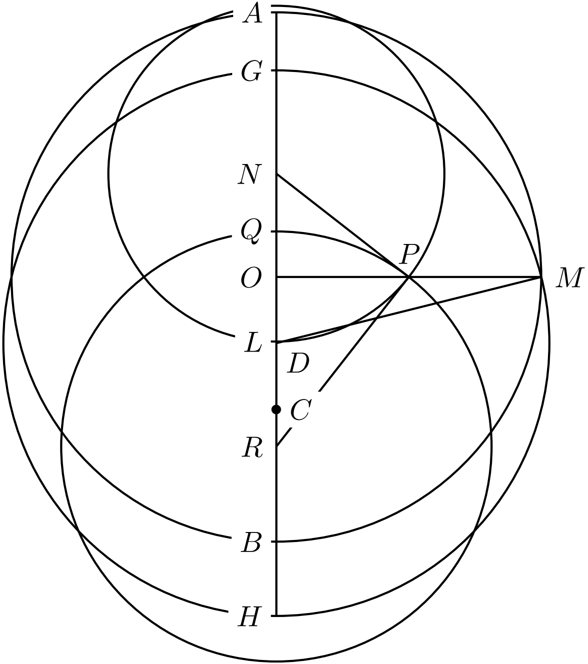


We join NP and RP.



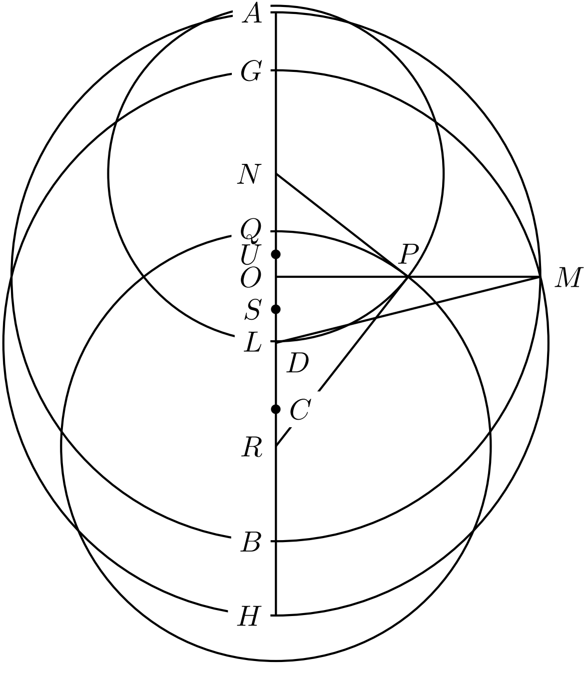
We construct circles

- with center N and distance NP, meeting NB at L (which is between O and D, though this is hard to see);
- with center R and distance RP, meeting RA at Q.

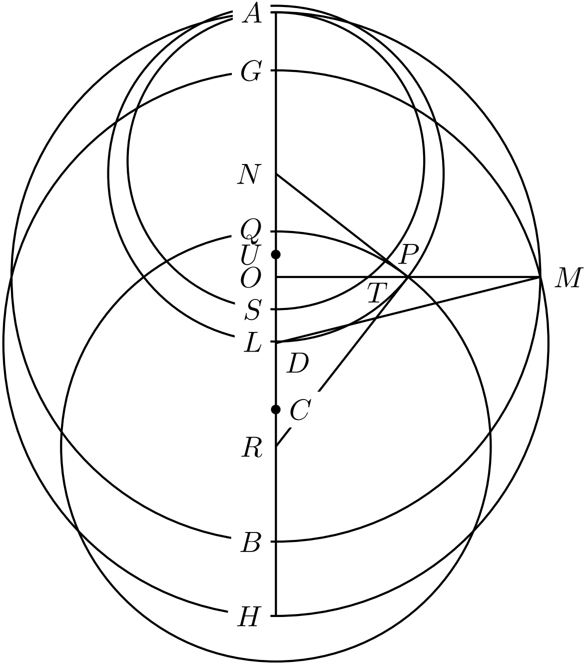


We let the midpoint of

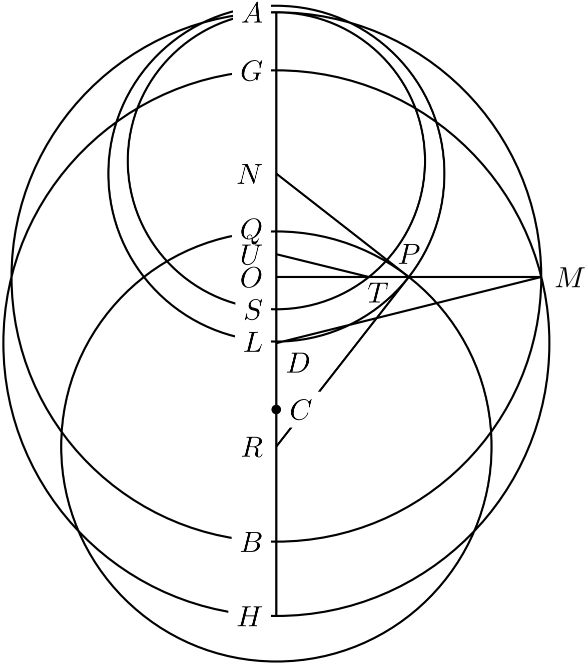
- OL be S,
- OQ be U.



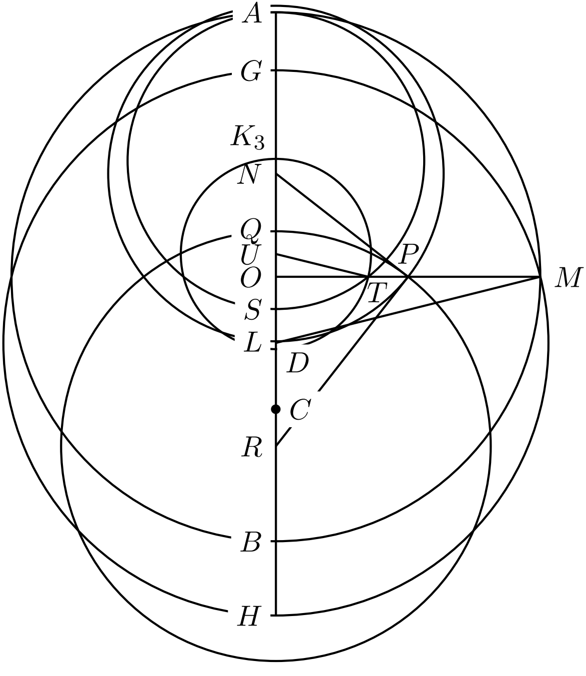
Now an asymmetry: We let the circle with diameter AS meet OM at T.



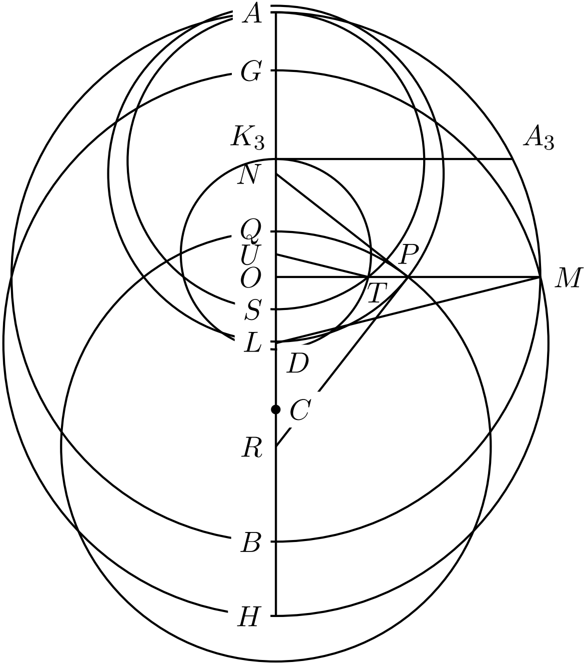
We join UT.



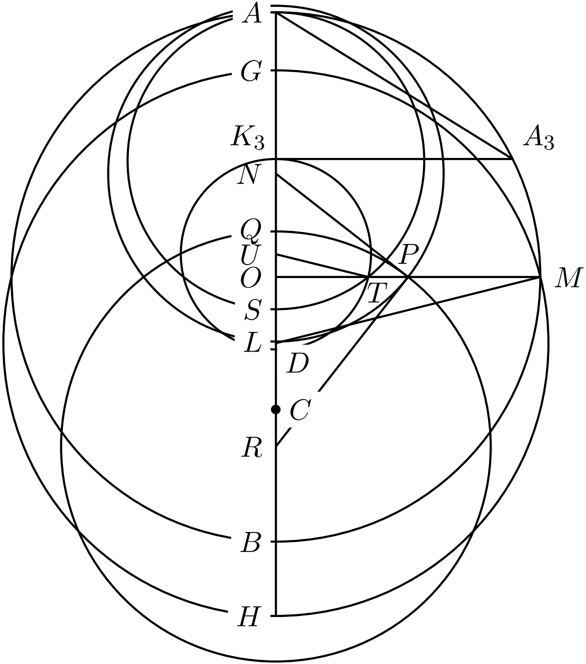
We construct the circle with center U and distance UT, meeting UA at K_3.



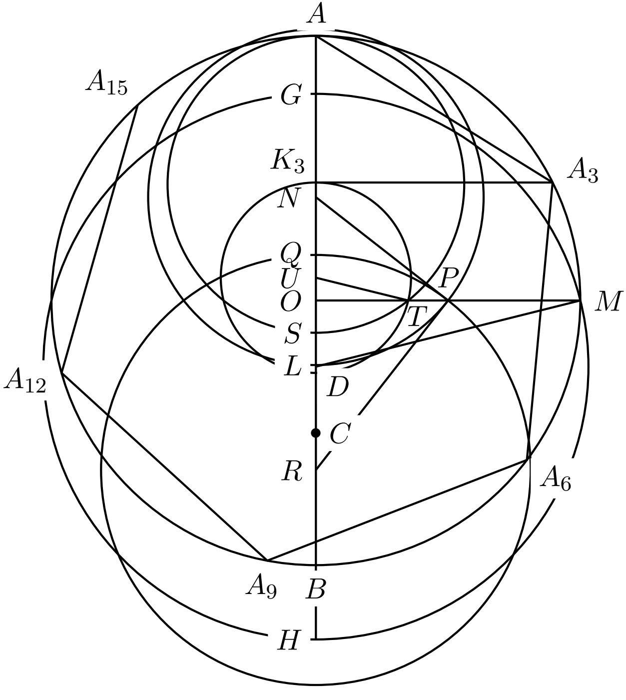
We erect a perpendicular to AB at K_3, meeting the original circle at A_3.



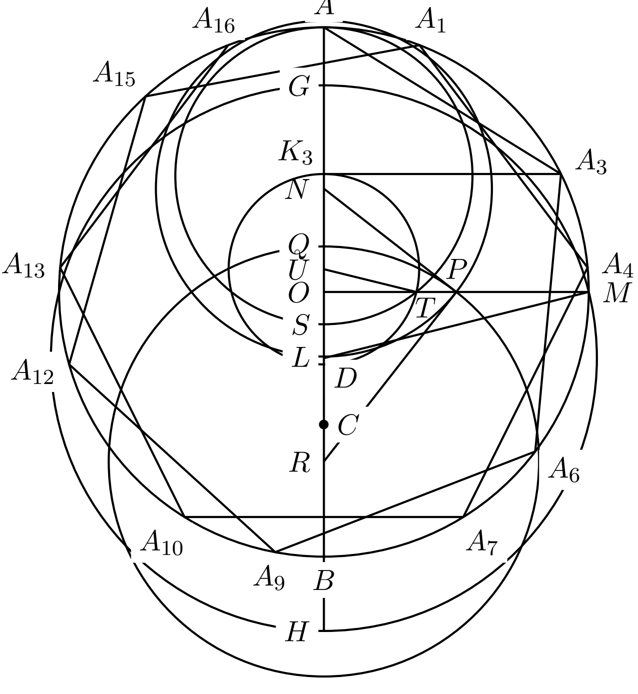
We join AA_3.



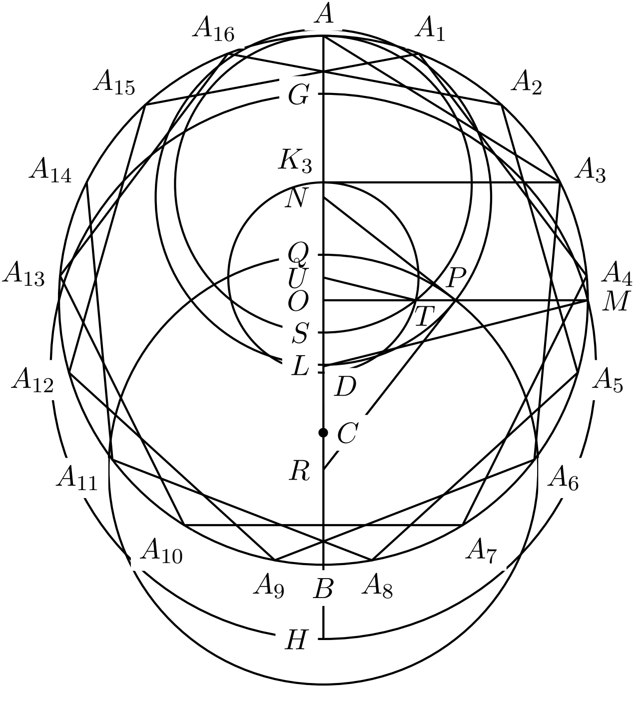
By Proposition IV.23 of the *Elements,* we fit in the original circle, end to end, five lengths equal to AA_3.



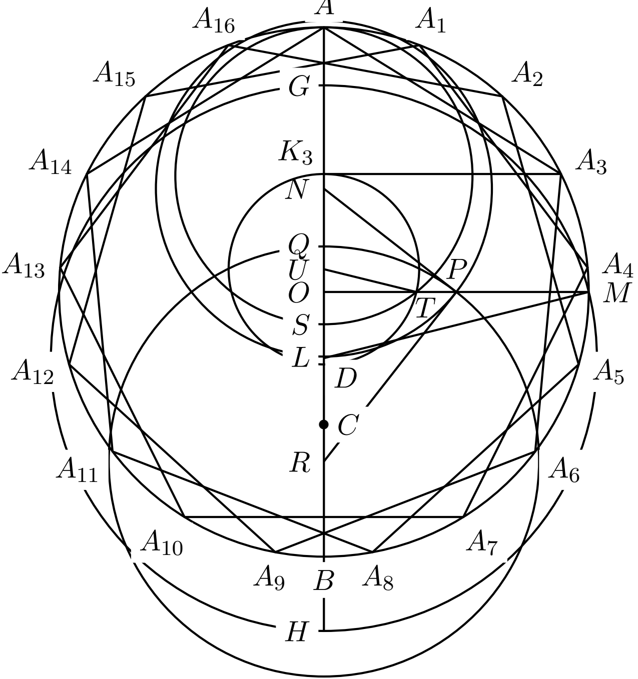
We fit in six more lengths equal to AA_3.



We fit in five more lengths equal to AA_3.



Finally, we join $A_{14}A$. This too is equal to AA_3, so that we have constructed the seventeen vertices of a regular heptakaidecagon; but we must show this.



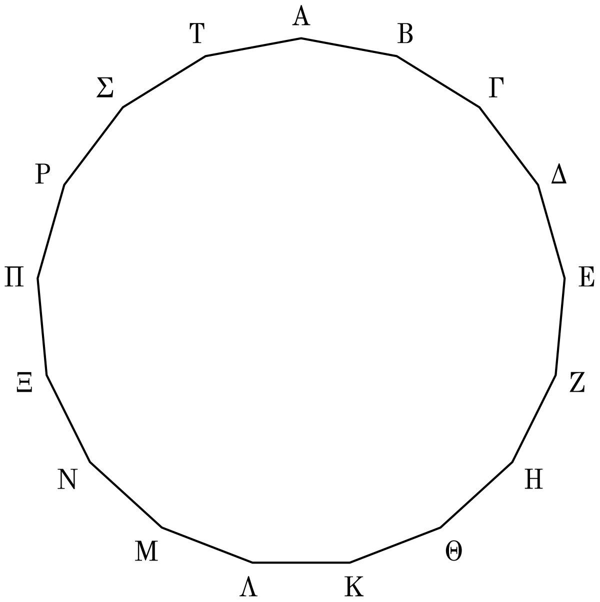
3 Analysis

Assuming a regular heptakaidekagon exists, we ask what we can use to

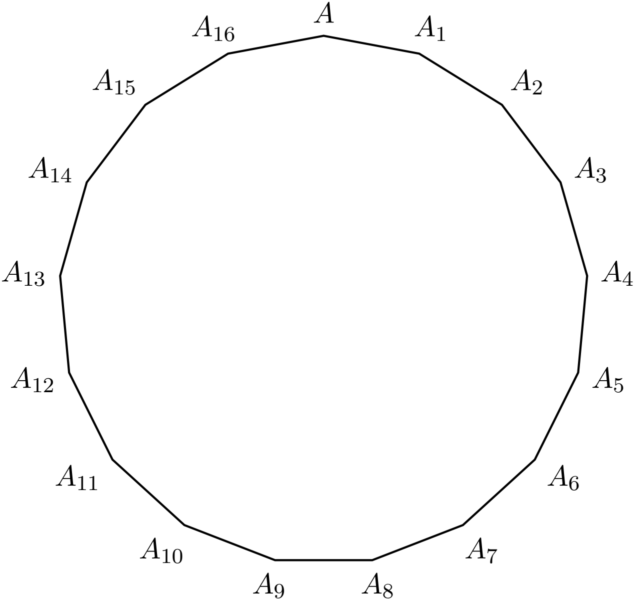
- construct it, then
- prove that we have constructed it.

We shall try to do everything in Euclidean terms.

However, our work will be easier if we use the modern practice of *numbering* the vertices, instead of just lettering them.



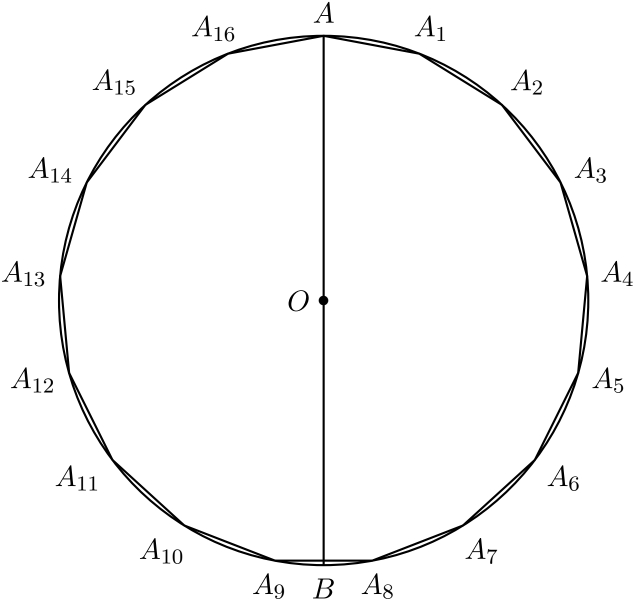
We number the vertices as if on a clock with seventeen hours.



We intend for our heptakaidekagon to be inscribed in a circle whose

- center is O,
- diameter is AB.

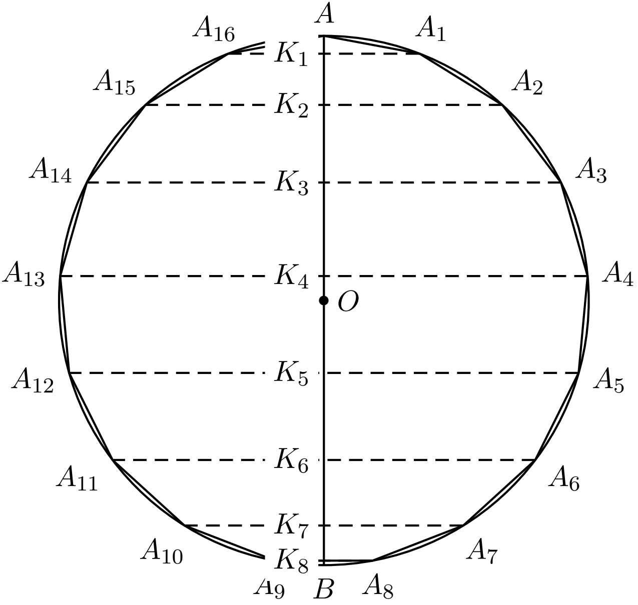
Then O is the center of gravity of the heptakaidekagon, by such arguments as Archimedes makes in *De planorum aequilibriis sive de centris gravitatis I* [2, 3].



We let the foot on AB of the perpendicular from

- A_1 be K_1,
- A_2 be K_2,

and so on to K_8, which is also the foot of the perpendicular from K_9, as K_7 is, from K_{10}, and so on.
If we can construct any of the points K_1, ..., K_8, then we can construct the heptakaidekagon.



Given two points X and Y on AB, we can find Z on AB meeting two conditions:

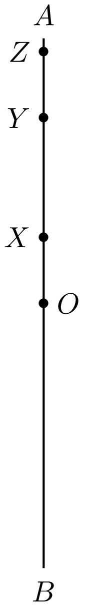
1. $YZ = OX$.
2. Z and X are on the same sides of Y and O respectively.

If the first condition is satisfied, the second condition is equivalent to

$$XZ = OY.$$

If both conditions are satisfied, let us say Z is the **sum** of X and Y, and let us write

$$Z = X + Y.$$

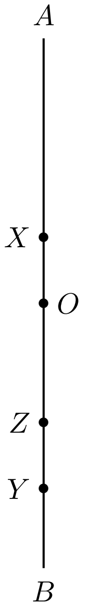


The two points X and Y need not be on the same side of O. In any case, again, if

$$X + Y = Z,$$

this means

$$OX = YZ,$$
$$OY = XZ.$$



Perhaps adding two points together is a pretty modern idea. However, the equation $X + Y = Z$ abbreviates

$$OX + OY = OZ,$$

but here OX, OY, and OZ are not just *segments,* but *directed segments.*

We can even allow Y not to be in a straight line with O and X. In this case, we still say

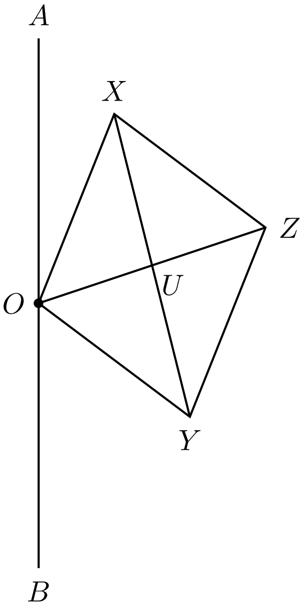
$$X + Y = Z,$$

provided $OXZY$ is a parallelogram. As Archimedes shows, the center of gravity of the parallelogram is the intersection of the two diagonals. If this is U, then also

$$Z = 2U,$$

and therefore

$$X + Y = 2U.$$



We now have

$$A + B = O,$$
$$A_1 + A_{16} = 2K_1,$$
$$A_2 + A_{15} = 2K_2,$$

and so on. By symmetry,

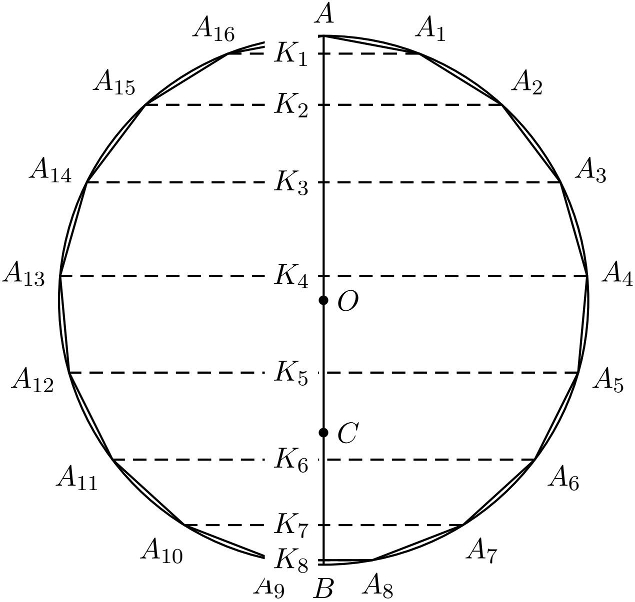
$$A + A_1 + \cdots + A_{16} = O.$$

Hence

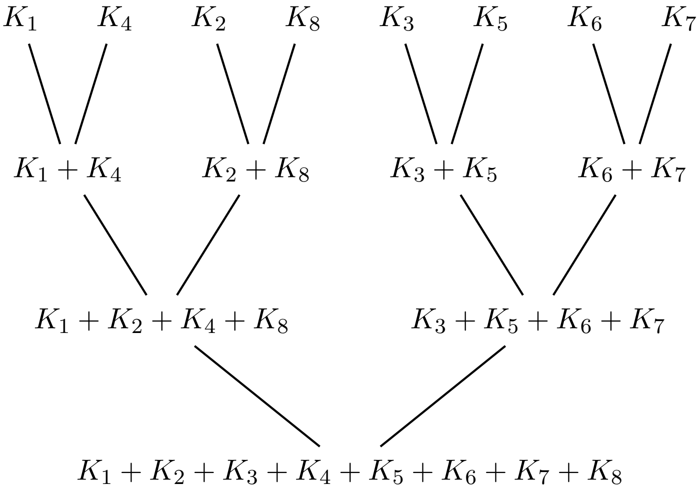
$$B = A_1 + \cdots + A_{16}$$
$$= 2(K_1 + \cdots + K_8).$$

Finally, C being the mid-point of OB,

$$C = K_1 + \cdots + K_8.$$



Now C is the sum at the root of the tree below. We are going to construct the rest of these sums, from root to branch.



We start by letting

$$K_1 + K_2 + K_4 + K_8 = G,$$
$$K_3 + K_5 + K_6 + K_7 = H.$$

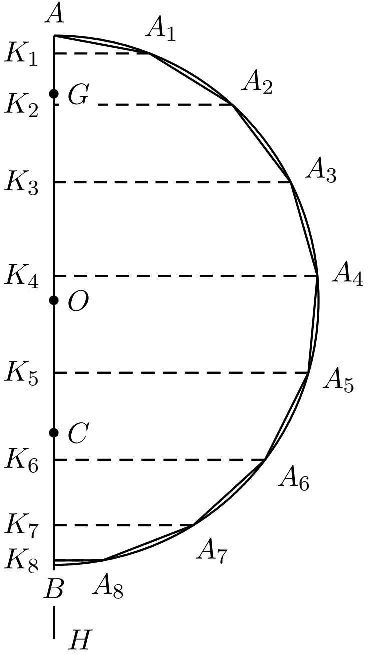
Then

$$G + H = C.$$

We shall show how

$$G \cdot H = A \cdot B,$$

and this will let us construct G and H.



Of two points X and Y on AB, the **product,**

$$X \cdot Y,$$

is a rectangle whose sides are equal to OX and OY; also, if

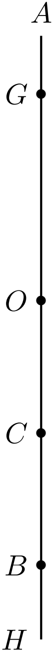
$$X \cdot Y = Z \cdot W,$$

this means X and Y, and Z and W, are alike on the same side of O, or opposite sides.

Again, we are going to show

$$G \cdot H = A \cdot B.$$

We shall do this, using **Ptolemy's Theorem,** from the *Almagest* [10, pages 16–7], [11, pages 50–1], [13, pages 422–5].



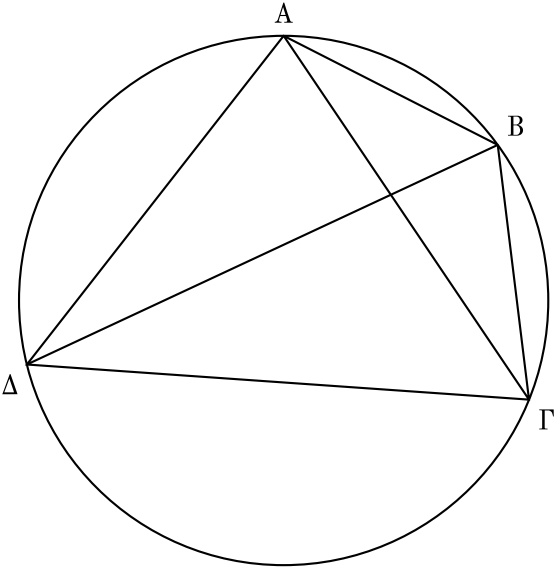
Ptolemy's Theorem is that, if a quadrilateral ΑΒΓΔ is inscribed in a circle, then

$$\mathrm{A\Gamma} \cdot \mathrm{B\Delta} = \mathrm{AB} \cdot \Gamma\Delta + \mathrm{B\Gamma} \cdot \mathrm{A\Delta}$$

—the product of the diagonals is the sum of the products of the opposite sides.

We shall prove this using

- similar triangles, as in Book VI of the *Elements*,
- Proposition III.26 ("The angles in the same segment are equal to one another").



In particular,

$$\angle \mathrm{A\Gamma B} = \angle \mathrm{A\Delta B}.$$

When E on BΔ ensures

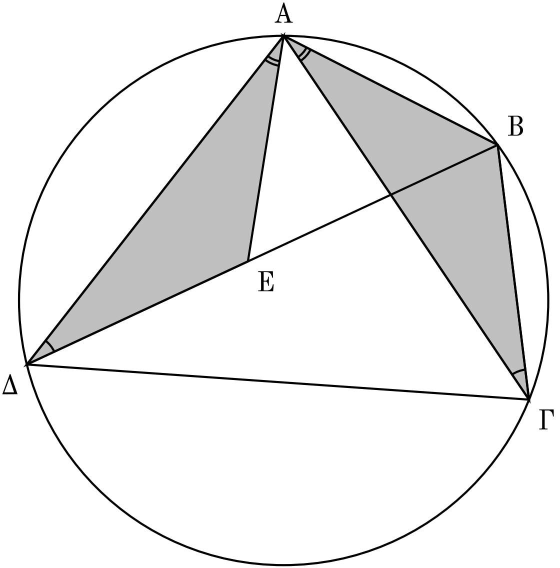
$$\angle \mathrm{\Gamma AB} = \angle \mathrm{\Delta AE},$$

then triangles ΓAB and ΔAE are similar by Proposition VI.4, and thus

$$\mathrm{A\Gamma} : \mathrm{B\Gamma} :: \mathrm{A\Delta} : \mathrm{E\Delta}.$$

By Proposition VI.16 then,

$$\mathrm{A\Gamma} \cdot \mathrm{E\Delta} = \mathrm{A\Delta} \cdot \mathrm{B\Gamma}.$$



Again,

$$A\Gamma \cdot E\Delta = A\Delta \cdot B\Gamma.$$

Likewise, ΓΔA and BEA are similar, so

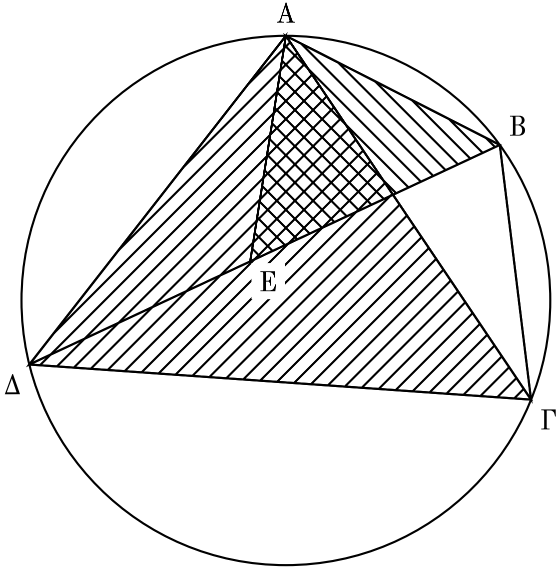
$$A\Gamma \cdot BE = AB \cdot \Gamma\Delta.$$

By Proposition II.1,

$$A\Gamma \cdot B\Delta = A\Gamma \cdot BE + A\Gamma \cdot E\Delta.$$

All this combined gives

$$A\Gamma \cdot B\Delta = AB \cdot \Gamma\Delta + B\Gamma \cdot A\Delta.$$



Thus, Ptolemy's Theorem is that a certain product is a sum of two products. We shall give those two products a common factor.

On a circle with diameter OA,

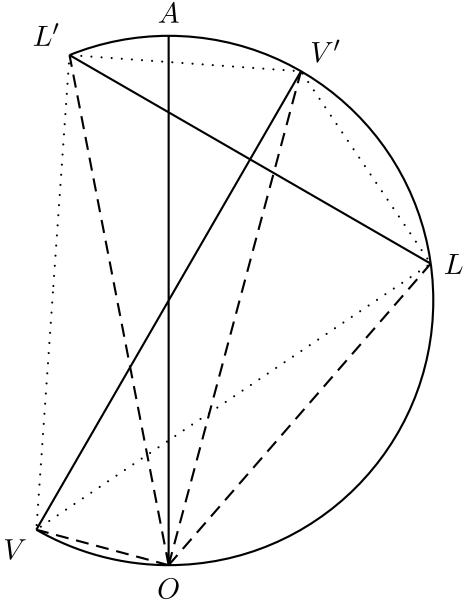
- A chord VV' is at right angles to a diameter VV', so that, by Proposition III.3 of the *Elements,*

$$VL = VL', \quad V'L = V'L'.$$

- O is between V and L on the circumference.

We shall show

$$2OV' \cdot VL = (OL' + OL) \cdot OA,$$
$$2OV \cdot V'L = (OL' - OL) \cdot OA.$$

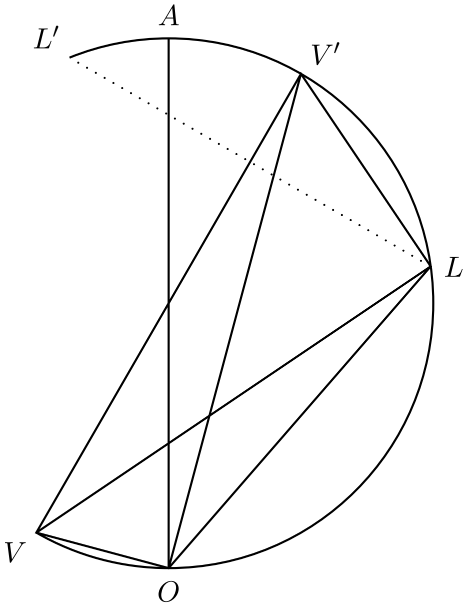


We first look at $OLV'V$, in which

$$OV' \cdot VL = OV \cdot V'L + OL \cdot VV'.$$

As Ptolemy observes, since VV' is a diameter,

$$OV' \cdot VL = OV \cdot V'L + OL \cdot OA.$$

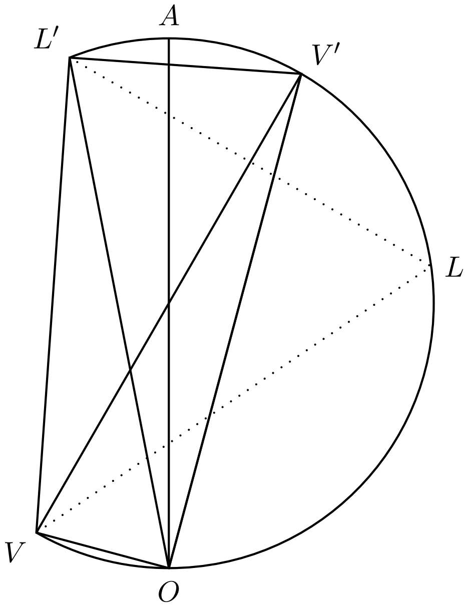


Likewise, in $OV'L'V$,

$$OV' \cdot VL' + OV \cdot V'L' = OL' \cdot VV',$$

and therefore

$$OV' \cdot VL + OV \cdot V'L = OL' \cdot OA.$$

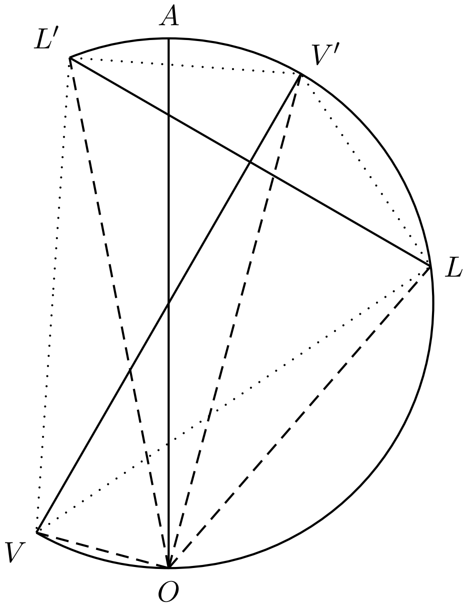


Our two results are

$$OV' \cdot VL + OV \cdot V'L = OL' \cdot OA,$$
$$OV' \cdot VL = OV \cdot V'L + OL \cdot OA.$$

Adding and subtracting them respectively yields the **geometric product formulas**

$$\begin{aligned} 2OV' \cdot VL &= OL' \cdot OA + OL \cdot OA, \\ &= (OL' + OL) \cdot OA, \\ 2OV \cdot V'L &= OL' \cdot OA - OL \cdot OA, \\ &= (OL' - OL) \cdot OA. \end{aligned}$$



Our first formula,

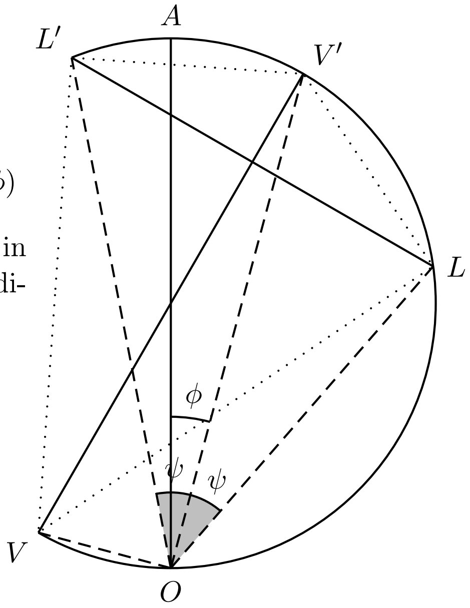
$$2OV' \cdot VL = (OL' + OL) \cdot OA,$$

takes the modern form

$$2\cos\phi \cdot \cos\psi = \cos(\psi - \phi) + \cos(\psi + \phi)$$

under the following correspondence in right triangles whose hypotenuses are diameters.

length	arc	angle
OV'	AV'	ϕ
VL	$V'L$	ψ
OL'	$L'A$	$\psi - \phi$
OL	AL	$\psi + \phi$



We have just established the *product formula for cosines,*

$$2\cos\phi\cdot\cos\psi = \cos(\psi-\phi)+\cos(\psi+\phi),$$

when ϕ and ψ are acute. Since

$$\cos(-\phi) = \cos\phi, \qquad\qquad \cos(\pi-\phi) = -\cos\phi,$$

if ϕ is obtuse, then

$$2\cos(\pi-\phi)\cdot\cos\psi = \cos(\psi-\pi+\phi)+\cos(\psi+\pi-\phi)$$

and therefore the product formula still holds. We shall apply it, understanding the cosine of the angle AOA_ℓ to be the ratio of K_ℓ to A, so that

$$2K_\ell\cdot K_m = (K_{\ell+m}+K_{\ell-m})\cdot A.$$

Meanwhile, we can obtain the product formula for cosines, one angle obtuse, from the second of our geometric formulas.

Thus

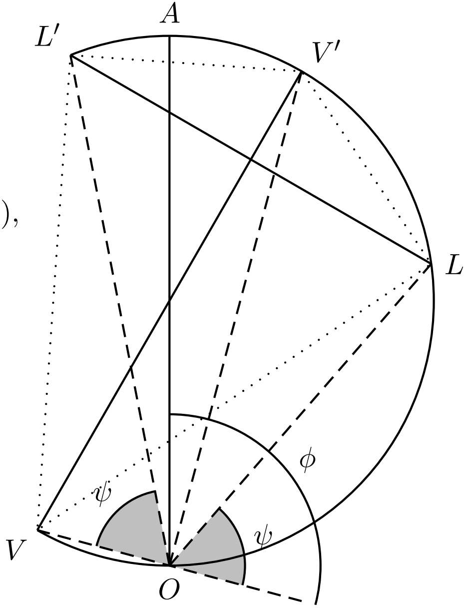
$$2OV \cdot V'L = (OL' - OL) \cdot OA$$

also becomes

$$2\cos\phi \cdot \cos\psi = \cos(\psi + \phi) + \cos(\psi - \phi),$$

but with ϕ obtuse, under:

length	arc	angle
OV	VA	$\pi - \phi$
$V'L$	VL'	ψ
OL'	$L'A$	$\pi - \phi - \psi$
OL	AL	$\phi - \psi$



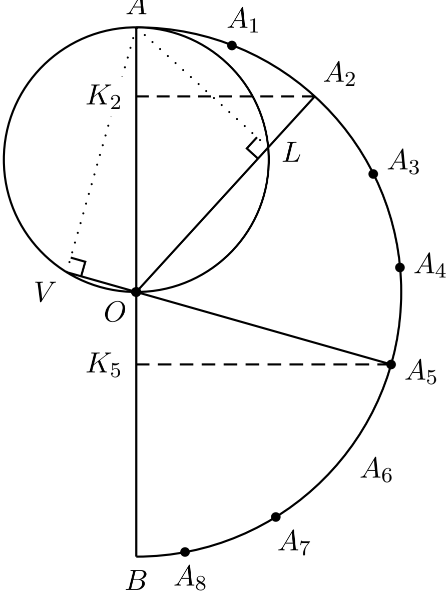
We now establish geometrically a special case of the rule

$$2K_\ell \cdot K_m = (K_{\ell+m} + K_{\ell-m}) \cdot A.$$

One can also just skip ahead to page 57.

We may choose V and L so that, for example,

$$OL = OK_2, \qquad OV = OK_5.$$



Since VV' is a diameter, also OV' is at right angles to OA_5.

Since, as angles,

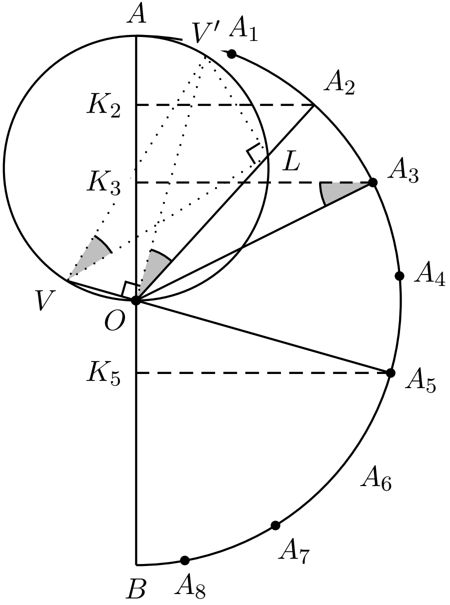
$$K_3OA_3 = A_2OA_5,$$

their complements are equal, and therefore

$$OA_3K_3 = V'OL = V'VL.$$

Consequently

$$V'L = OK_3.$$



Finally, since OV' bisects LOL', also OA_5 bisects the angle *externally.*

At the same time, OA_5 bisects A_2OA_8. Thus L' lies on the extension of A_8O and

$$OL' = OK_8.$$

Now

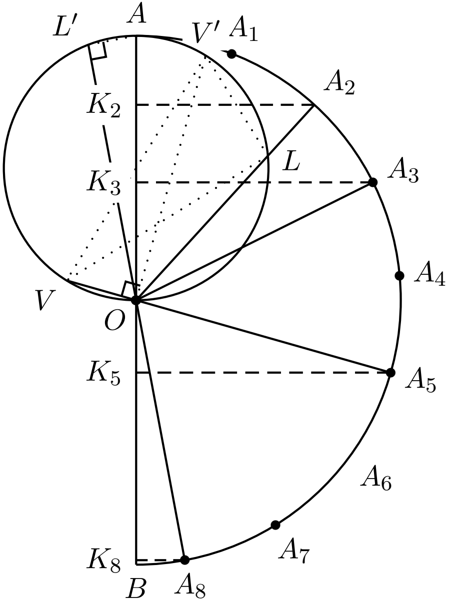
$$2OV \cdot V'L = (OL' - OL) \cdot OA$$

becomes

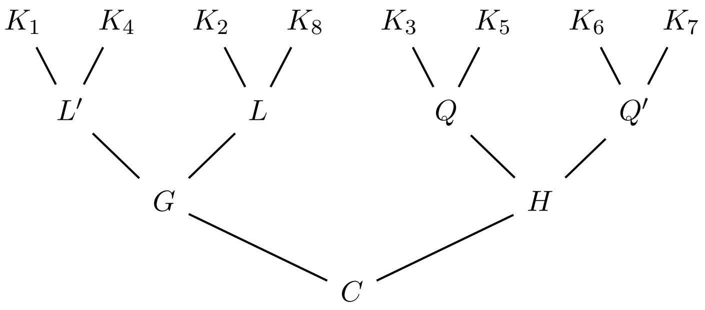
$$2OK_5 \cdot OK_3 = (OK_8 - OK_2) \cdot OA,$$

and therefore

$$2K_3 \cdot K_5 = (K_2 + K_8) \cdot A.$$



Our tree from earlier will be



where again the node just below a pair of nodes is their sum. We shall be showing

$$G \cdot H = B \cdot A, \qquad L' \cdot L = D \cdot A, \qquad Q \cdot Q' = D \cdot A.$$

This is somehow ensured by the analysis of the points $K_1, \ldots, K_8$ given by the tree. The general idea is as follows, but one also can skip it and go to page 64.

We said the points $A_1, A_2, \ldots$ were numbered as if on a clock with 17 hours. Thus we can understand

$$A_1 = A_{18} = A_{35} = A_{52} = \ldots$$

and likewise for A_2, A_3, and so on. Also,

$$A_1 = A_{-16} = A_{-33} = A_{-50} = \ldots$$

In particular,

$$2K_1 = A_1 + A_{16} = A_1 + A_{-1}, \qquad 2K_2 = A_2 + A_{15} = A_2 + A_{-2},$$

and so on.

The last proposition about numbers in the *Elements,* namely Proposition IX.36, has the enunciation, in Heath's translation,

> If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

Thus we are to set out

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128, \quad 256,$$

and so on, and the successive sums of the terms are

$$1, \quad 3, \quad 7, \quad 15, \quad 31, \quad 63, \quad 127, \quad 255, \quad 511.$$

Since 3, 7, 31, and 127 are prime, each of the products $2 \cdot 3$, $4 \cdot 7$, $16 \cdot 31$, and $64 \cdot 127$ is *perfect* in the sense of being "equal to its own parts," as in the definition at the head of Book VII; for example,

$$4 \cdot 7 = 28 = 1 + 2 + 4 + 7 + 14.$$

In the sequence

$$2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128, \quad 256$$

of powers of 2, the fourth, namely 16, being one *less* than 17, the eighth, namely 256, must be one *more* than a multiple of 17. Indeed,

$$256 = 17 \cdot 15 + 1.$$

No previous power has this property. Also,

$$6^2 = 36 = 17 \cdot 2 + 2.$$

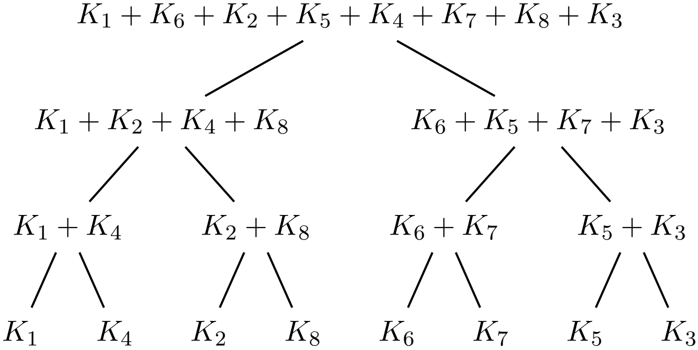
Therefore the first 16 powers of of 6 serve as indices for each of $A_1, \dots, A_{16}$. Here are the remainders of those powers after being measured or "divided" by 17.

6	6^2	6^3	6^4	6^5	6^6	6^7	6^8	6^9	6^{10}	6^{11}	6^{12}	6^{13}	6^{14}	6^{15}	6^{16}
6	2	12	4	7	8	14	16	11	15	5	13	10	9	3	1

In Gauss's terminology, the 16 different points A_k compose a *period* of length 16. Then, if we take every *other* point of a period, according to the order

$$1, \quad 6, \quad 2, \quad 12, \quad 4, \quad 7, \quad 8, \quad 14, \quad 16, \quad 11, \quad 15, \quad 5, \quad 13, \quad 10, \quad 9, \quad 3,$$

from the table (the number 1 can come first or last, it doesn't matter), this too is a period, of half the length. The halves of the sums of the points of the periods are just the nodes on our tree:



We can define the product of two points on the circle by the rule

$$A_\ell \cdot A_m = A_{\ell+m} \cdot A.$$

This is compatible with multiplication of points on AB; for example,

$$\begin{aligned} 2K_\ell \cdot 2K_m &= (A_\ell + A_{-\ell})(A_m + A_{-m}) \\ &= (A_{\ell+m} + A_{-\ell-m} + A_{\ell-m} + A_{-\ell+m}) \cdot A \\ &= 2(K_{\ell+m} + K_{\ell-m}) \cdot A, \end{aligned}$$

and thus

$$2K_\ell \cdot K_m = (K_{\ell+m} + K_{\ell-m}) \cdot A.$$

This is a case of what Gauss shows, that (A being treated as 1) *the product of the sums of two periods of the same length is a sum of that many periods of that length.* For example, writing A_m as $[m]$ as Gauss does, we have

$$2(K_2 + K_8) \cdot 2(K_5 + K_3) = ([2] + [8] + [15] + [9]) \cdot ([5] + [3] + [12] + [14]).$$

We compute as on the next page.

$$
\begin{aligned}
&([2]+[8]+[15]+[9])\cdot([5]+[3]+[12]+[14])\\
&=([2]+[2\cdot 6^4]+[2\cdot 6^8]+[2\cdot 6^{12}])\cdot([5]+[5\cdot 6^4]+[5\cdot 6^8]+[5\cdot 6^{12}]),
\end{aligned}
$$

and this is

$$
\begin{aligned}
&[2+5]+[2+5\cdot 6^4]+[2+5\cdot 6^8]+[2+5\cdot 6^{12}]\\
&+[2\cdot 6^4+5]+[2\cdot 6^4+5\cdot 6^4]+[2\cdot 6^4+5\cdot 6^8]+[2\cdot 6^4+5\cdot 6^{12}]\\
&+[2\cdot 6^8+5]+[2\cdot 6^8+5\cdot 6^4]+[2\cdot 6^8+5\cdot 6^8]+[2\cdot 6^8+5\cdot 6^{12}]\\
&+[2\cdot 6^{12}+5]+[2\cdot 6^{12}+5\cdot 6^4]+[2\cdot 6^{12}+5\cdot 6^8]+[2\cdot 6^{12}+5\cdot 6^{12}].
\end{aligned}
$$

If we rearrange the summands of each row, we get

$$
\begin{aligned}
&[2+5]+[2+5\cdot 6^4]+[2+5\cdot 6^8]+[2+5\cdot 6^{12}]\\
&+[2\cdot 6^4+5\cdot 6^4]+[2\cdot 6^4+5\cdot 6^8]+[2\cdot 6^4+5\cdot 6^{12}]+[2\cdot 6^4+5]\\
&+[2\cdot 6^8+5\cdot 6^8]+[2\cdot 6^8+5\cdot 6^{12}]+[2\cdot 6^8+5]+[2\cdot 6^8+5\cdot 6^4]\\
&+[2\cdot 6^{12}+5\cdot 6^{12}]+[2\cdot 6^{12}+5]+[2\cdot 6^{12}+5\cdot 6^4]+[2\cdot 6^{12}+5\cdot 6^8].
\end{aligned}
$$

Each column is a sum $[k]+[k\cdot 6^4]+[k\cdot 6^8]+[k\cdot 6^{12}]$ (since $[6^{16}]=[1]$).

Starting out with

$$G = K_1 + K_2 + K_4 + K_8,$$
$$H = K_3 + K_5 + K_6 + K_7.$$

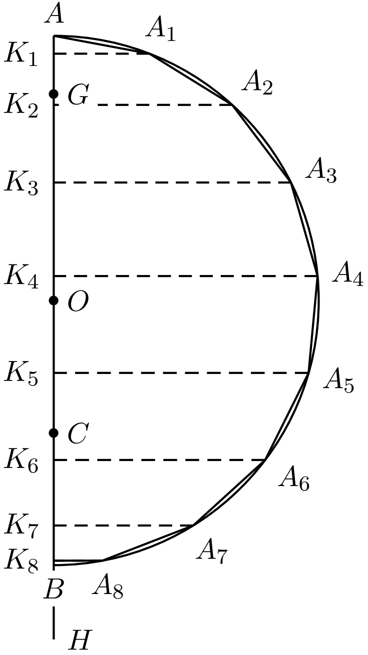
We shall construct G and H from

- $G + H$, which is C;
- $G \cdot H$, which is a sum of sixteeen products $K_\ell \cdot K_m$, each of which is half of $(K_{\ell+m} + K_{\ell-m}) \cdot A$.

Again, we shall end up with

$$G \cdot H = A \cdot B,$$

and then we can find G and H using Proposition II.6 of the *Elements.*



We compute

$$
\begin{aligned}
2K_1 \cdot K_3 &= (K_2 + K_4) \cdot A, & 2K_2 \cdot K_3 &= (K_1 + K_5) \cdot A,\\
2K_1 \cdot K_5 &= (K_4 + K_6) \cdot A, & 2K_2 \cdot K_5 &= (K_3 + K_7) \cdot A,\\
2K_1 \cdot K_6 &= (K_5 + K_7) \cdot A, & 2K_2 \cdot K_6 &= (K_4 + K_8) \cdot A,\\
2K_1 \cdot K_7 &= (K_6 + K_8) \cdot A, & 2K_2 \cdot K_7 &= (K_5 + K_8) \cdot A,
\end{aligned}
$$

$$
\begin{aligned}
2K_4 \cdot K_3 &= (K_1 + K_7) \cdot A, & 2K_8 \cdot K_3 &= (K_5 + K_6) \cdot A,\\
2K_4 \cdot K_5 &= (K_1 + K_8) \cdot A, & 2K_8 \cdot K_5 &= (K_3 + K_4) \cdot A,\\
2K_4 \cdot K_6 &= (K_2 + K_7) \cdot A, & 2K_8 \cdot K_6 &= (K_2 + K_3) \cdot A,\\
2K_4 \cdot K_7 &= (K_3 + K_6) \cdot A, & 2K_8 \cdot K_7 &= (K_1 + K_2) \cdot A.
\end{aligned}
$$

Each of $K_1, \ldots, K_8$ appears four times on the right, so

$$
\begin{aligned}
G \cdot H = (K_1 + K_2 + K_4 + K_8) \cdot (K_3 + K_5 + K_6 + K_7)&\\
= 2(K_1 + \cdots + K_8) \cdot A = 2C \cdot A = A \cdot B.&
\end{aligned}
$$

By Proposition II.6 of the *Elements,* if

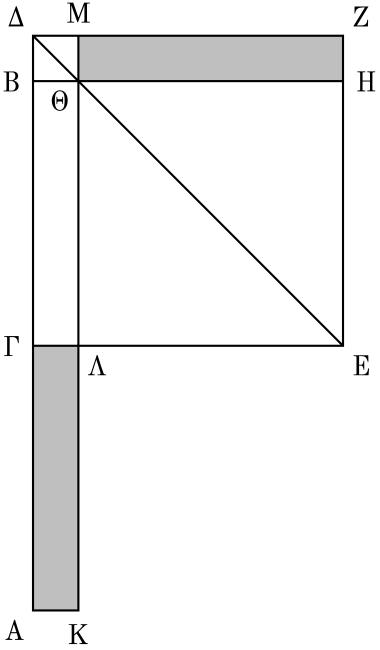
- Γ bisects AB,
- BΔ is added to AB,

then

$$A\Delta \cdot B\Delta + \Gamma B^2 = \Gamma\Delta^2,$$

that is, equal are

- the sum of
  - the rectangle contained by
    - the whole AB with the added length BΔ, and
    - the added length BΔ; and
  - the square on the half, namely ΓB;
- the square on ΓΔ, which is the half with the added length.



Again, $\mathrm{A\Gamma} = \mathrm{\Gamma B}$ and

$$\mathrm{A\Delta} \cdot \mathrm{B\Delta} + \mathrm{\Gamma B}^2 = \mathrm{\Gamma\Delta}^2.$$

Perhaps we are given

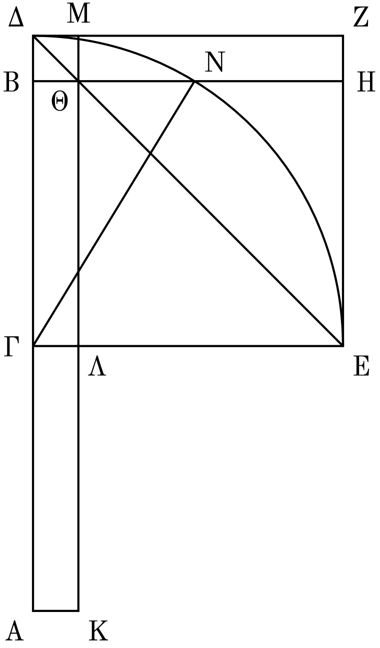
- AB, thus Γ too;
- not BΔ itself;
- but the product $\mathrm{A\Delta} \cdot \mathrm{B\Delta}$, as the square on BN, this being at right angles to AB.

Then by Proposition I.47,

$$\mathrm{BN}^2 + \mathrm{\Gamma B}^2 = \mathrm{\Gamma N}^2.$$

Now we can find Δ, lying on the extension of AB so that

$$\mathrm{\Gamma\Delta} = \mathrm{\Gamma N}.$$



In short, given AB and BN, we have found Δ on the extension of AB so that

$$\mathrm{A\Delta} \cdot \mathrm{B\Delta} = \mathrm{BN}^2.$$

That Δ lies on the extension of AB means

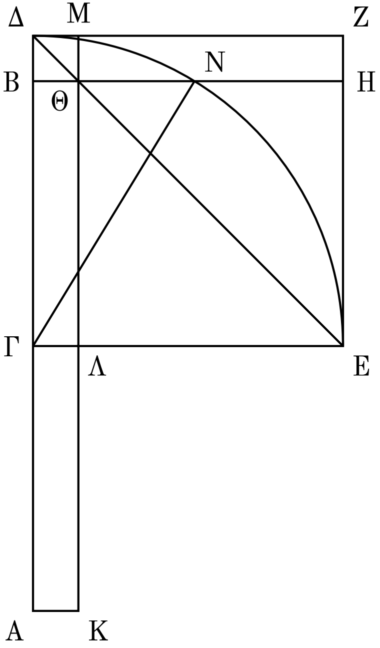
$$\mathrm{A\Delta} = \mathrm{AB} + \mathrm{B\Delta}.$$

Though Euclid does not distinguish between BΔ and ΔB, we do, writing the last equation also as

$$\mathrm{A\Delta} + \mathrm{\Delta B} = \mathrm{AB}.$$

Thus we have found AΔ and ΔB, given

- their sum, AB;
- their product, BN^2.



What we have done in Cartesian terms is show

$$a, b = \frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - ab},$$

at least when

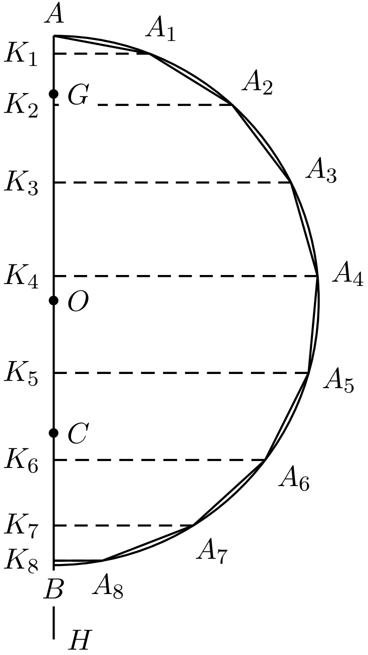
$$b < 0 < a.$$

Meanwhile, on our diameter AB, we found for G and H the conditions

$$G + H = C, \qquad G \cdot H = A \cdot B.$$

In more traditional terms, the conditions are

$$GC = OH, \qquad OG \cdot CG = OA^2.$$



Those conditions again are

$$GC = OH, \qquad OG \cdot CG = OA^2,$$

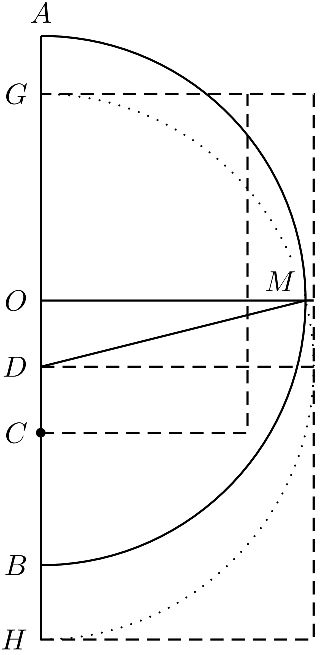
and from them we find G and H by first letting the perpendicular to AB at O meet our circle at M, so that

$$OM = OA.$$

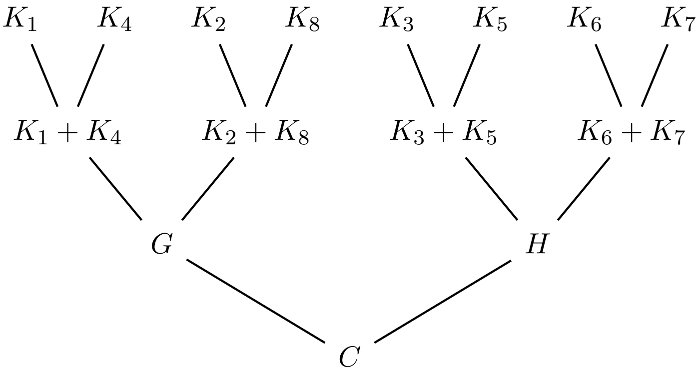
Since also D is the midpoint of OC, the second condition above becomes

$$OG \cdot CG = OM^2,$$
$$OG \cdot CG + OD^2 = OM^2 + OD^2,$$
$$DG^2 = DM^2,$$

and thus G and H are on the circle through M with center D.



The tree we made earlier is now as follows.



We have constructed C, G, and H. We can obtain K_3 and K_5, thus the whole heptakaidecagon, from $K_3 + K_5$ and $K_3 \cdot K_5$, and therefore, since

$$2K_3 \cdot K_5 = (K_2 + K_8) \cdot A,$$

from $K_3 + K_5$ and $K_2 + K_8$. Now we are going to obtain *these* two sums.

We seek L and L', where

$$L = K_2 + K_8, \qquad L' = K_1 + K_4,$$

so

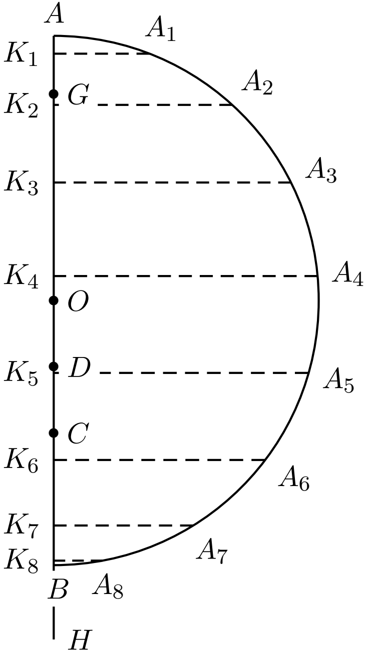
$$L + L' = G;$$

also, L will be on the B side of O, L' on the A side. We compute

$$L \cdot L' = D \cdot A,$$

since $2D = C = K_1 + \cdots + K_8$ and

$$2K_2 \cdot K_1 = (K_1 + K_3) \cdot A,$$
$$2K_2 \cdot K_4 = (K_2 + K_6) \cdot A,$$
$$2K_8 \cdot K_1 = (K_7 + K_8) \cdot A,$$
$$2K_8 \cdot K_4 = (K_4 + K_5) \cdot A.$$



Thus

$$L + L' = G, \qquad L \cdot L' = D \cdot A,$$

that is,

$$LG = OL', \qquad OL \cdot GL = OC^2.$$

Then L and L' are the intersections with AB of the circle with center the midpoint, N, of OG, passing through the midpoint, P, of OM. Again,

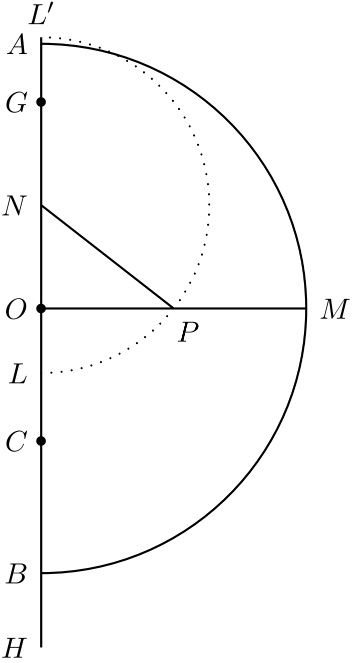
$$L = K_2 + K_8,$$

so

$$2K_3 \cdot K_5 = L \cdot A.$$

We now find

$$K_3 + K_5.$$



We seek Q and Q', where

$$Q = K_3 + K_5, \qquad Q' = K_6 + K_7,$$

so

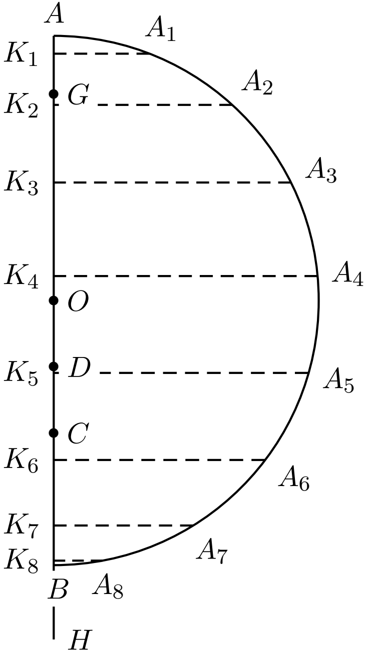
$$Q + Q' = H;$$

also, Q will be on the A side of O, Q' on the B side. We compute

$$Q \cdot Q' = D \cdot A,$$

since $2D = C = K_1 + \cdots + K_8$ and

$$2K_3 \cdot K_6 = (K_3 + K_8) \cdot A,$$
$$2K_3 \cdot K_7 = (K_4 + K_7) \cdot A,$$
$$2K_5 \cdot K_6 = (K_1 + K_6) \cdot A,$$
$$2K_5 \cdot K_7 = (K_2 + K_5) \cdot A.$$



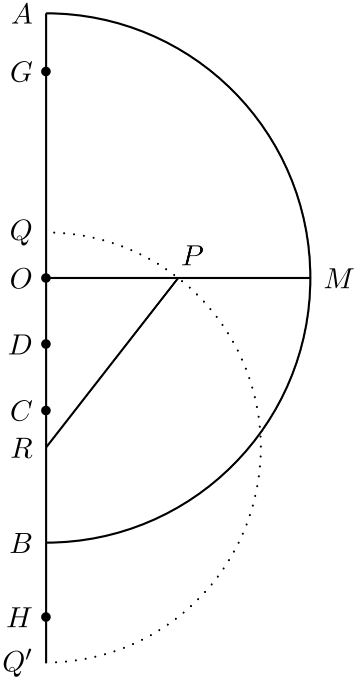
Thus

$$Q + Q' = H, \qquad Q \cdot Q' = D \cdot A,$$

that is,

$$QH = OQ', \qquad OQ \cdot HQ = OC^2.$$

Then Q and Q' are the intersections with AB of the circle passing through P and with center the midpoint, R, of OH.



Finally, letting

$$2S = L = K_2 + K_8,$$

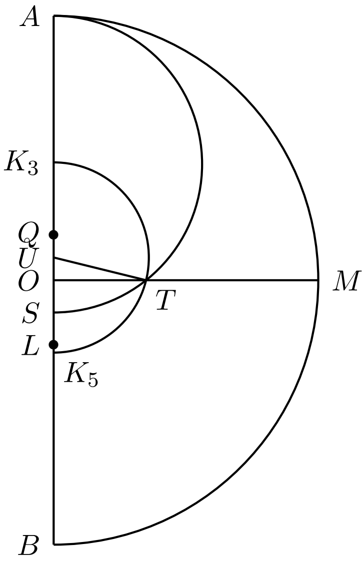
we have

$$K_3 + K_5 = Q, \qquad K_3 \cdot K_5 = S \cdot A,$$

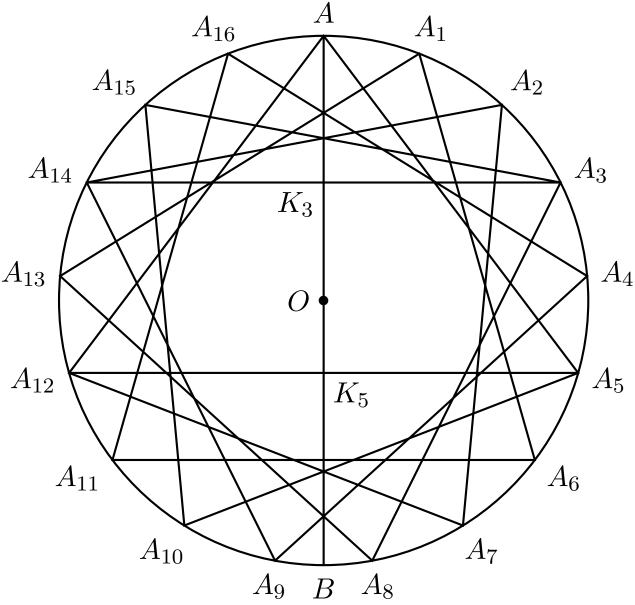
that is,

$$K_3Q = OK_5, \qquad OK_3 \cdot QK_3 = OT^2,$$

where, by Proposition II.14, T is the intersection with OM of the circle with diameter SA. Then K_3 and K_5 are the intersections with AB of the circle passing through T whose center is the midpoint U of OQ.



We now obtain A_3 and A_5, and the heptakaidecagon from either of these: here, from K_5.



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