On Gödel's Incompleteness Theorem

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January 1, 2025

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Summary

Gödel's Incompleteness Theorem is about the logic of mathematics. It is that a certain mathematical structure is so rich that its theory cannot be completely axiomatized. This means there will always be true statements about the structure that cannot be proved as theorems from previously given axioms. To give meaning to this conclusion, we review some examples of mathematical theorems, and their proofs, in geometry, algebra, and logic; we also give an example of a structure that is so simple (while still being interesting) that its theory *can* be

completely axiomatized. First we look at a couple of popular descriptions of Gödel's Theorem; these can be misleading. We pass to Raymond Smullyan's interpretation of Gödel's theorem as a puzzle; then to an analogy with the incompleteness of an English guide to English style. Gödel's argument relies on converting statements about numbers into numbers themselves; we note how to argue similarly by understanding geometrical statements as geometrical diagrams. Geometry is thus somehow incomplete; likewise, physics.

Contents

1 Introduction

The purpose of this essay is to give some meaning to the following statement of Gödel's Incompleteness Theorem: *In a* *certain logical language developed from symbols for addition and multiplication, there can be no method for writing down statements from which we can prove everything that is true of the counting numbers, but nothing that is false.*

We are going to talk a lot about statements. A statement is a declarative sentence, if you like, or whatever it is that such a sentence expresses; we expect it to be true or false, but not both. Simple examples of mathematical statements include $1 + 1 = 2$ and $1 + 1 = 0$: each of these is true in the right context. A theorem is a statement that has been proved to be true.

Gödel's theorem is peculiar for being a statement *about* certain statements. Briefly, the theorem is that we cannot *axiomatize* the complete theory of the counting numbers as equipped with addition and multiplication.

By contrast, we *can* axiomatize the complete theories of many other structures. The part of mathematical logic called model theory is based on this fact; indeed, one of its early texts (from 1956) is Abraham Robinson's *Complete Theories* [23], which gives a number of examples of what the title names.

The point is left out of popular accounts of Gödel's theorem. For example, in a 2023 interview [5], Andrew Granville says,

> To discuss mathematics, you need a language, and a set of rules to follow in that language. In the 1930s, Gödel proved that no matter how you select your language, there are always statements in that language that are true but that can't be proved from your starting axioms.

On the contrary, it *does* matter how you select your language. Before Gödel proved what he did, Presburger gave a complete axiomatization of the integers as equipped with addition alone

[27, 21]. The language of addition alone is not strong *enough* to keep us from writing down the axioms of a complete theory of the integers (or just the positive integers, namely the counting numbers). For a simpler example in the same language, we shall show, in § 6, the completeness of the theory of the infinite sequences of ones and minus-ones, where the "sum" of two sequences is the sequence of the products of corresponding entries.

Possibly Granville means to acknowledge that such examples exist when he continues:

> It's actually more complicated than that, but still, you have this philosophical dilemma immediately: What is a true statement if you can't justify it? It's crazy.

There may be crazy things in the world; however, Gödel's Theorem *is* a justification of the true statement in question. Granville goes on:

> So there's a big mess. We are limited in what we can do.

We are limited by *ignorance* in what we can do; Gödel's Theorem, like every other theorem, is not a limitation, but an *example* of what we can do.

Raymond Smullyan has the Supreme Being make the general point in the dialogue called "Is God a Taoist?" [25]:

> GOD: My dear fellow, I could no more choose to give you free will than I could choose to make an equilateral triangle equiangular. I could choose to make or not to make an equilateral triangle in the

> first place, but having chosen to make one, I would then have no choice but to make it equiangular.
>
> MORTAL: I thought you could do anything!
>
> GOD: Only things which are logically possible. As St. Thomas said, "It is a sin to regard the fact that God cannot do the impossible, as a limitation on His powers." I agree, except that in place of his using the word *sin* I would use the term *error.*

Smullyan does not give a source for his quotation, which may not be authentic. Something that Thomas Aquinas *does* say seems closer—because less judgmental—to what Smullyan's God would say [20, *ST*, I, Q. 25, Art. 3, p. 231]:

> Therefore, everything that does not imply a contradiction in terms is numbered among those possibles in respect of which God is called omnipotent: whereas whatever implies contradiction does not come within the scope of divine omnipotence, because it cannot have the aspect of possibility. Hence it is more appropriate to say that such things cannot be done, than that God cannot do them.

Lauren Burns writes of things that cannot be done, and yet goes on to do them anyway. This is in *Triple Helix,* a book of 2022 that is explained by its subtitle, "My donor-conceived story" [3, ch. 3, p. 22]:

> In 1931 Austrian mathematician Kurt Gödel published his famous incompleteness theorems. The work shattered deeply held beliefs that logical systems offered the one true pathway to perfect, absolute truth. Gödel's astonishing proof revealed what

> others had intuited but nobody before him had been able to prove: that any formal mathematical system—no matter how powerful—is intrinsically incomplete. In other words, the only reality that we can hold on to is that some truths will always lie beyond the boundary of the verifiable. There exist questions that you just can't answer.

Normally a metaphor provides illumination, but what if itself is obscure? I would propose three corrections to Burns's account of Gödel's theorem. (1) No *one* logical system offers "the one true pathway to . . . truth," because (2) there is a system that is powerful *enough* to be incomplete, and thus (3) some truths cannot be verified, some questions cannot be answered, *in that system.*

By referring to Gödel's incompleteness *theorems,* in the plural, Burns may be alluding to the "first" and "second" ones, to be distinguished below in § 4.

Meanwhile, Burns's questions concern who she is, once she learns that it was not her father who supplied the sperm that fertilized the egg that grew into her, physically speaking:

> A little over a year since Mum's confession, on the surface everything was normal. Our family life continued as before. I was doing well in my studies in aerospace engineering. But below the surface lurked things I kept to myself. Everything had changed. Deep within my own private truth of unprovable feelings I felt like Gödel's daughter; intrinsically incomplete.

As an adoptee myself, I enjoyed Burns's book, which I learned about from an interview in the *Guardian* [11]. There, another of Burns's metaphors is quoted:

> the desert is most shaped by the thing it lacks—water. Much of my life had been shaped by what was missing from it, too.

Burns elaborates on her next page (which is 23):

> In the place where I had inherited half my genes all I could see was a void. By extension I felt part of that void, hollow and empty.

Her feeling like Gödel's daughter does not keep Burns from answering the question of where the other half of her genes came from. She achieves *that* kind of completeness, at least.

Before he proved the Incompleteness Theorem, Gödel himself proved a completeness theorem, whereby every statement that is "always" true can be proved [12]. We shall look at this in § 7. As for the general notion of mathematical proof, in § 5 we shall look at its first use historically, in geometry. None of this will draw on any *particular* mathematical knowledge beyond high-school algebra. In fact we shall look some algebraic proofs in § 6.

One should understand that, for example, if $f(x)$ or simply f stands for the polynomial $ax^2 + bx + c$, then $f(t)$ stands for $at^2 + bt + c$, and if this evaluates to 0, then t is a solution of the equation $f = 0$. One should understand such things, even if the Latin letters are replaced with such Greek minuscule letters as α, β, θ, σ, ϕ, χ, or ψ (alpha, beta, theta, sigma, phi, chi, or psi). All of the Greek capitals will label points in Figure 3.

We outline a proof of Gödel's Incompleteness Theorem in § 3, with a bit more detail in §4. There we also introduce the metaphor or analogy of a book *about* English, *in* English.

Meanwhile, § 2 is a second introduction. Actually, §§ 5–7 could be read as an introduction, or at least as a warning: without knowledge of such mathematics as is in them, one may misunderstand Gödel's theorem.

2 Puzzling

Gödel's Incompleteness Theorem, or more precisely one proof of it, is based on a variant of the so-called Liar Paradox, which can be traced to the Epistle of Paul to Titus in the Greek Bible. Titus was "ordained the first bishop of the church of the Cretians," according to a codicil given at the end of the epistle in the King James Version, first published in 1611 [4, NT pp. 265–7]. In the first of the epistle's three chapters, Paul writes,

> 12 One of themselves, *even* a prophet of their own, said, The Cretians *are* alway [*sic*] liars, evil beasts, slow bellies.
>
> 13 This witness is true. Wherefore rebuke them sharply, that they may be sound in the faith . . .

In the original, Paul's quotation of the Cretan prophet reads,

Κρήτες αεί ψεῦσται, κακά θηρία, γαστέρες άργαί.

This is the first of the Presocratic fragments that Diels and Kranz attribute to Epimenides [7, 3B1, p. 32].

That a Cretan should assert that Cretans are always liars: one may take this for an absurdity or a paradox. Interpreted strictly, the assertion becomes a puzzle: is it true or false? We assume it is one or the other, but not both. If it were true, then, as the word of a Cretan, it would be false. Therefore, from the two Biblical verses above, by strict logic we can conclude:

1. It is false that all Cretans are liars.
2. Some Cretan must not be a liar.
3. The Cretan prophet is a liar.
4. Paul is a liar.

Though not mentioning Paul, Raymond Smullyan draws the first three of the four conclusions above in his book called *What Is the Name of This Book? The Riddle of Dracula and Other Logical Puzzles* [26, ¶ 253, p. 214].

I was intrigued, as a child, by a review of Smullyan's book in Martin Gardner's "Mathematical Games" column [10, ch. 20, pp. 281–92]. A few years later, I found a used copy of the book itself in a junk shop. I enjoyed the book, except the sixteenth and final chapter, which was impenetrable. The chapter was an exposition of Gödel's Incompleteness Theorem.

Briefly, Gödel's theorem is that, in certain mathematical systems, there is no way to ensure that all true statements are theorems. Later we shall look at some examples of mathematical systems, in which there will be such true statements as,

> an exterior angle of a triangle is greater than an opposite interior angle;
>
> zero plus anything is that thing;
>
> every statement entails itself.

For now, all we need know about a mathematical system is that it provides a way to write down statements and, designating some of the true ones as axioms, prove others as theorems.

If we know a statement of a system is true, we must have *some* kind of proof for it. Gödel shows that, in some systems, no matter what axioms we designate, there will always be a true statement that is not a theorem. If we take that statement as a new axiom, this will let us identify a new statement that is still not a theorem.

We cannot, at the outset, just designate all true statements as axioms. We have no way to do this, once for all. Indeed, this is one way to put Gödel's theorem.

Paradoxically, a true statement can fail to be a theorem, not because a system is too *weak* to prove it, but because the system is so strong as to let the statement be formulated in the first place. The system lets us make so *many* statements that some of them inevitably end up being unprovable, even though they are true.

Some of the unprovable statements are true, precisely because they cannot be proved. A mathematical system may let us formulate the statement, "I am not a theorem that can be proved from such-and-such axioms." This, like the assertion of Epimenides, is true or false, but not both. If it were false, then it *would* be a theorem, and theorems are true. In short, even if the statement were false, it would be true. Thus it must be true, but then, because of what it says, it cannot be a theorem.

That was an exposition of Gödel's Incompleteness Theorem. Such an exposition could take one of three forms:

- an annotation of Gödel's 1931 paper, "On formally undecidable propositions of *Principia mathematica* and related systems I" [13];
- a revision of that paper at the same or a higher level of detail and rigor;
- a summary treatment, leaving out details.

My exposition above was of the last type. Smullyan gives such an exposition too. Before giving more detail about Gödel's theorem, along with some simple examples of geometric, algebraic, and logical systems and their theorems, I want to give an exposition of *Smullyan's* exposition (Gardner does this too).

3 Smullyan's Account

Smullyan asks us conceive of two infinite lists [26, ¶ 269, p. 234–8]:

- a list A_1, A_2, A_3, ... of sets of counting numbers;
- a list σ_1, σ_2, σ_3, ... of statements of some mathematical system.

We are to make two suppositions about the system and the lists. Unfortunately the second supposition is even more complicated, logically, than the definition of "limit" at the heart of calculus:

1. *One* of the listed sets comprises the serial numbers of the listed statements that are *not* theorems of the system.
2. For *every* listed set—call it A—, there is *some* listed set B for which, for *every* number n, there is *some* number k for which
    - n is in B if and only if k is in A,
    - σ_k is the statement that n is in A_n.

We are going infer from the latter supposition a simpler statement. First of all, we can write the two bulleted conclusions of the second supposition more compactly as

$$\left.\begin{array}{c} n \in B \Leftrightarrow k \in A, \\ \sigma_k \text{ is the statement } n \in A_n. \end{array}\right\} \quad (1)$$

For every listed set A then, since the set B that is guaranteed by the second supposition is also listed, this set is, for some n, the set A_n. For this particular n, for some k, we can rewrite (1) as

$$n \in A_n \Leftrightarrow k \in A,$$
$$\sigma_k \text{ is the statement } n \in A_n.$$

Combining these, we can eliminate the mentions of n; thus, for every listed set A, for some k,

$$\sigma_k \Leftrightarrow k \in A.$$

This holds in particular if A is the set mentioned in the first supposition, namely the set of numbers of statements that are not theorems. If now k is not in A, that is, $k \notin A$, then σ_k must be a theorem, and therefore true; but then also, according to this theorem, $k \in A$. In short, $k \notin A$ implies $k \in A$. Therefore indeed $k \in A$. This means both that σ_k is true, and that it is not a theorem.

We can understand σ_k as saying that its own serial number belongs to the set of serial numbers of statements that are not theorems; in short, "I am not a theorem."

That is Gödel's basic result, as derived from Smullyan's two suppositions. For Smullyan, establishing the first of these "is quite a lengthy affair, though elementary in principle"; moreover, the second "is really a very simple matter." Achieving that simplicity still takes some work, which so far I have glossed over.

Smullyan first presents the listed statements metaphorically, as if they are either knights, who always speak the truth, or else knaves, who always lie. The listed sets are clubs of knights and knaves. I'm not sure that the metaphor provides clarity. To puzzle things out, I have introduced notation as above.

As any club has a criterion for membership, so any of the listed sets will have a definition, which can be written down. Our list of sets then is effectively a list of definitions of sets. We have to construct this list so that, given any number n, we have a mathematical way of inferring what the definition of A_n is. Now we can form the statement $n \in A_n$. We still need to assign to this statement a serial number from which alone we can infer the meaning of the statement. If we can do all of that, then, given a listed set A, we can define a set B consisting of all n such that the serial number of the statement $n \in A_n$ is itself in A. Finally, we have to be able to include B among the listed sets. This gives us the second supposition.

In the translation between listed sets and their serial numbers, mathematics talks about itself. I am going to liken this to how a grammar *of* English, written *in* English, implicitly talks about *itself.*

Smullyan gives little idea of what it means to be a mathematical theorem. I'm not sure one can have any real understanding of Gödel's theorem without knowing about mathematical proof. Thus I shall also work through some examples, from geometry, algebra, and logic itself.

One may conclude from Gödel's theorem that *mathematics* will always be incomplete. This does not mean that no particular mathematical system can be complete. I shall give some examples that are.

4 The Grammar of Gödel

Gödel's theorem is a *logical* theorem about proving *mathematical* theorems. Again, the theorem is that there are mathematical systems so strong as to contain a statement σ whose meaning is precisely that of the statement, "σ is not a theorem." In short then, σ is the statement, "I am not a theorem."

Mathematical statements are not normally in the first person; they do not feature anything like the pronoun *I*. For the effect of such a pronoun, Gödel makes use of the ambiguity alluded to in the second paragraph of § 1. A statement can be either of the following:

- something stated—call it a *meaning*;
- something spoken, or written down.

The spoken or written thing is a string of symbols, be they sounds, words, or letters; they all come from a catalogue of some kind—a phonology, a dictionary, an alphabet—and are put together according to certain grammatical rules.

A book such as Fowler's *Dictionary of Modern English Usage* [9] is *about* statements that are in English. It also *consists* of such statements, and is thus about itself, even though it may never literally refer to itself.

Fowler's book does in fact refer to itself, in the dedication to the memory of the author's brother,

> who shared with me the planning of this book, but did not live to share the writing.

There is also a more subtle reference, in the body of the *Dictionary,* where the entry headed **Inversion** continues,

> By this is meant the abandonment of the usual English order & the placing of the subject after the verb as in *Said he,* or after the auxiliary of the verb as in *What did he say?* & *Never shall we see his like again.*

In addition to the three explicit examples, the whole sentence is an example of inversion, since in the usual English order the sentence would read, "By this, the abandonment of the usual English order … is meant," or else "The abandonment … is meant by this."

In his *Dictionary,* Fowler has included inversion because, although it has its legitimate uses,

> the abuse of it ranks with Elegant Variation as one of the most repellent vices of modern writing.

Nonetheless, Fowler has already used inversion to start another entry, **Battered ornaments:**

> On this rubbish-heap are thrown, usually by a bare cross-reference, such synonyms of the Elegant Variation kind as *alma mater, daughter of Eve, sleep of the just,* & *brother of the* Angle …

Thus, were the article **Inversion** deleted from the dictionary, the remainder could be counted as incomplete, for using an example of a construction that readers should be wary of abusing in their own writing.

No account of good English will ever be strictly complete, since a poet can always come along, to write verses such as

> My father moved through dooms of love
> through sames of am through haves of give,
> singing each morning out of each night
> my father moved through depths of height

—That is E. E. Cummings [1, p. 256], who breaks the rules in a way that is still recognizably English and even good English.

Gödel is the Cummings of mathematics, writing down new true statements that cannot be proved by the old rules.

In a sufficiently rich mathematical system, we can give each statement a number—its *Gödel* number—in such a way that, for any number n, we can make the statement,

Statement number n is not a theorem.

let us call *this* statement

$$\phi(n).$$

Showing how to write this down is what is "quite a lengthy affair, though elementary in principle," in the words of Smullyan, already quoted in the previous section.

We can define Gödel numbers themselves briefly. Each of the symbols that are used to write down statements is assigned a number. For example, to work with numbers in base ten, we might assign each of the ten digits to itself, then number other symbols as follows.

symbol	$=$	$+$	$\times$	(	)
number	10	11	12	13	14

We can use our new dictionary to convert the statement

$$3 \times (4 + 8) = 36$$

into the sequence

$$(3, 12, 13, 4, 11, 8, 14, 10, 3, 6)$$

of numbers. We then form this into the single number

$$2^3 \cdot 3^{12} \cdot 5^{13} \cdot 7^4 \cdot 11^{11} \cdot 13^8 \cdot 17^{14} \cdot 19^{10} \cdot 23^3 \cdot 29^6,$$

where the bases of the powers are the primes, ordered by size. The product of the powers is the Gödel number of the original statement. Because every number is uniquely the product of primes, we can recover every statement from its Gödel number.

Now it makes sense to talk about the set of Gödel numbers of statements in a given system that are not theorems of the system. If the system is *arithmetical,* in the sense of already being about numbers, then we may ask whether the set of Gödel numbers of the theorems (or non-theorems) is somehow definable. It could be the solution set of some equations and inequations, but there are other ways to define sets. Since we can understand a proof as a list of statements (and there are examples of such lists in §§ 5 and 7), we can convert the proof itself into a Gödel number. Thus the question of whether some statement is a theorem becomes the question of whether some kind of number exists, at least if that kind of number is itself definable.

We need not go into details. A system of arithmetic may be too weak to formulate the statement $\phi(n)$ above. Here n is a number, and the system is about numbers, but it cannot talk about whether n is the Gödel number of a theorem. This is one kind of "incompleteness," although the terminology is not usually used this way. In § 1, we mentioned Presburger arithmetic, where there is only addition. By Gödel's argument, this system must be weak in the sense now contemplated, because every true statement that *can* be formulated in terms of addition alone *is* a theorem of the system: again, in short, the system is complete.

A system with addition *and* multiplication is strong enough to formulate $\phi(n)$, as Gödel shows; but in any case, *if* we have $\phi(n)$, we obtain it from some other expression, ϕ, by replacing a variable with n. Let us denote the Gödel number of any expression θ by $\ulcorner\theta\urcorner$ (this notation is not Gödel's, but was used in 1973 by Shoenfield [24, § 6.6, p. 122], having been introduced in 1940 by Quine for another purpose [22, ch. 1, § 6, p. 35]). Gödel shows how to solve the equation

$$\ulcorner\phi(n)\urcorner = n. \tag{2}$$

We shall not need to know more than that there *is* a solution; thus one can skip the next paragraph.

Smullyan's exposition shows how a solution of (2) is obtained. Each listed set A is, for some one-variable expression θ, the set $\{x\colon \theta(x)\}$, namely the set of numbers x for which $\theta(x)$ is true. We let the serial number of A be $\ulcorner\theta\urcorner$. Likewise, we let the serial number of every listed statement σ be $\ulcorner\sigma\urcorner$. Thus we can replace Smullyan's list of sets with a list of one-variable expressions. The first supposition is now that this list includes ϕ. To express the second supposition, instead of the letters A, B, n, and k, we use θ, ψ, and χ, according to the following dictionary.

A	B	n	σ_k
$\{x\colon \theta(x)\}$	$\{x\colon \psi(x)\}$	$\ulcorner\chi\urcorner$	$\chi(\ulcorner\chi\urcorner)$

Now the second supposition is that, for every listed expression θ, for some listed expression ψ, for every listed expression χ,

$$\psi(\ulcorner\chi\urcorner) \text{ is the statement } \theta(\ulcorner\chi(\ulcorner\chi\urcorner)\urcorner).$$

This is the new version of (1), by our dictionary. We can now let θ be ϕ, and then we can let χ be the resulting ψ, so that

$$\psi(\ulcorner\psi\urcorner) \text{ is the statement } \phi(\ulcorner\psi(\ulcorner\psi\urcorner)\urcorner).$$

This means the number $\ulcorner\psi(\ulcorner\psi\urcorner)\urcorner$ solves (2).

If some number a solves (2), so that $\phi(a)$ is statement number a, then this statement has the meaning of,

$$\phi(a) \text{ is not a theorem.}$$

Thus it means,

I am not a theorem.

We concluded earlier that such a statement must be true, on the basis that *theorems* are true. Thus the class of truths is strictly larger than the class of theorems. This is Gödel's *First* Incompleteness Theorem.

To prove that theorem, we need not actually know that all theorems are true; it is enough that no logical contradiction be a theorem. We can write down two statements,

$$\begin{gathered}\phi(a) \text{ is a theorem,}\\ \phi(a) \text{ is not a theorem,}\end{gathered}$$

which contradict one another. The latter statement being $\phi(a)$, if this were a theorem, then so would the former be; for, in our sufficiently rich system, if some statement is a theorem, then so is the *statement* that it is. Thus, if contradictions are not allowed, then $\phi(a)$ cannot be a theorem, and therefore it is true.

That contradictions cannot be proved is itself a statement σ. Now we can form the implication

$$\sigma \Rightarrow \phi(a),$$

namely "If σ is true, then so is $\phi(a)$," or "σ implies $\phi(a)$"; and we have shown that *this* is a theorem. Since $\phi(a)$ is *not* a theorem, it follows that σ cannot be a theorem, *if* it is true; for there is a *rule of inference* whereby any statement β must be a theorem, if there is a theorem α for which $\alpha \Rightarrow \beta$ is also a theorem.

In short, our system cannot prove that it is not contradictory, if indeed it is not. This is Gödel's *Second* Incompleteness Theorem.

In mathematics, one is well advised to rewrite what one reads in one's own words and symbolism. Smullyan did this for Gödel, and I did it for Smullyan. If people who do it for Gödel are computer scientists, such as Douglas R. Hofstadter in *Gödel, Escher, Bach* [16], then their accounts end up looking more or less odd to me as a mathematician. What I write about Gödel may likewise look odd to the non-mathematician—or not just odd, but lacking in attention to certain details. I want to pay some of that attention now.

In mathematics, as I have said, we prove theorems from axioms. We choose our axioms as we wish, but they should be true. More precisely, they should be true *in* or *of* some mathematical structure that we want to understand. In this case, the structure *satisfies* the axioms. In our research, we take up, as *hypotheses,* statements that we believe the structure to satisfy. Since we can be mistaken, we try to *prove* our hypotheses as theorems that follow from our axioms.

That is some kind of idealized picture, at least. Let us investigate how it works.

5 Geometry

I say our proofs are based on axioms. Not all mathematicians may be prepared to explain what axioms they are working with. "However," as Timothy Gowers observes in *Mathematics: A Very Short Introduction* [14, p. 41],

> if somebody makes an important claim and other mathematicians find it hard to follow the proof,

> they will ask for clarification, and the process will then begin of dividing steps of the proof into smaller, more easily understood substeps.

Those steps and substeps are ultimately based on axioms. As Gowers has observed on his previous page, the possibility of analyzing a proof into its smallest steps

> is far from obvious: in fact it was one of the great discoveries of the early 20th century, largely due to Frege, Russell, and Whitehead ... it means that *any dispute ahout the validity of a mathematical proof can always be resolved.*

If this is indeed a discovery, then it must be true; but I think it is true in the way that axioms are true. Axioms are true by *fiat.* That mathematical disputes can always be resolved is a *conviction* that guides us in our work. I think the conviction means mathematics is pacifist or rather *pacificist*—peace-making—, at least in principle; in practice, we may not always have the courage of the conviction.

The prototypical examples of mathematical axioms are the postulates of Euclid's *Elements.* The first four of these make possible the following activities.

1. To connect two points with a straight line.
2. To extend a given straight line as far as we like.
3. To draw a circle with any given center and radius.
4. To know that all right angles are equal to one another.

By the first three postulates, we have a ruler and compass, along with a flat surface to use them on, and an implement to leave marks or scratches with. Euclid's word *γραμμή* for line [8] is from the verb *γράφω* "to scratch" [2].

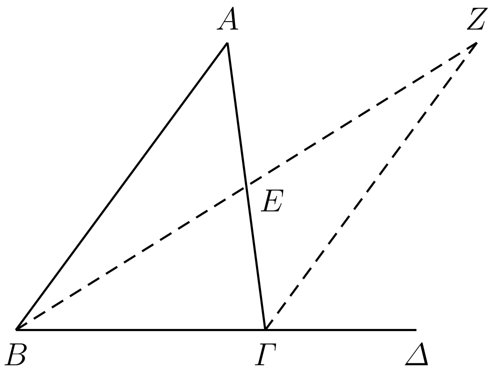

Figure 1: Euclid's Proposition I.16

In Postulate 4, Euclid gives us a set square. This is not to *draw* right angles with; Euclid will show how to do that with ruler and compass. The existence of a set square confirms that all right angles are indeed equal.

Euclid's first four postulates entail that in any triangle, the exterior angle at any vertex is greater than either of the two opposite interior angles. This is the 16th proposition of Book I of the *Elements.* Thus if triangle $AB\mathit{\Gamma}$ is given, and side $B\mathit{\Gamma}$ is extended to $\mathit{\Delta}$ as in Figure 1, then

$$\angle \mathit{\Delta\Gamma} A > \angle BA\mathit{\Gamma}.$$

This is a theorem. To prove it in Euclid's way, we complete the figure as follows.

1. Bisect $A\mathit{\Gamma}$ at E, using Proposition 10 from earlier in the *Elements,* so that
$$AE = \mathit{\Gamma} E.$$
2. Connect BE, using Postulate 1.

3. Extend that straight line to some point Z, using Postulate 2.
4. Adjust that point as needed, using Postulate 3, to ensure

$$BE = EZ.$$

5. Connect $Z\Gamma$, using Postulate 1 again.

The figure complete, we continue with Euclid's argument.

6. By Proposition 15, that vertical angles are equal,

$$\angle AEB = \angle \Gamma EZ.$$

7. From our three displayed equations, by Proposition 4, "Side Angle Side,"

$$\angle EAB = \angle E\Gamma Z.$$

8. However, by inspection,

$$\angle E\Gamma Z < \angle E\Gamma\Delta. \tag{3}$$

9. From this and the previous equation,

$$\angle EAB < \angle E\Gamma\Delta.$$

10. Since angles EAB and $E\Gamma\Delta$ are respectively ΓAB and $A\Gamma\Delta$, the desired conclusion follows.

Does it really follow? Various objections can be proposed. For example, how do we know that BE extends far enough to reach the desired point Z, which we do not know to exist before we find it? Also, how exactly do we know (3)?

Consider how we can draw triangles on a globe, letting "straight lines" be segments of great circles. For example, we can let A and B lie on the equator, while Γ is at the north

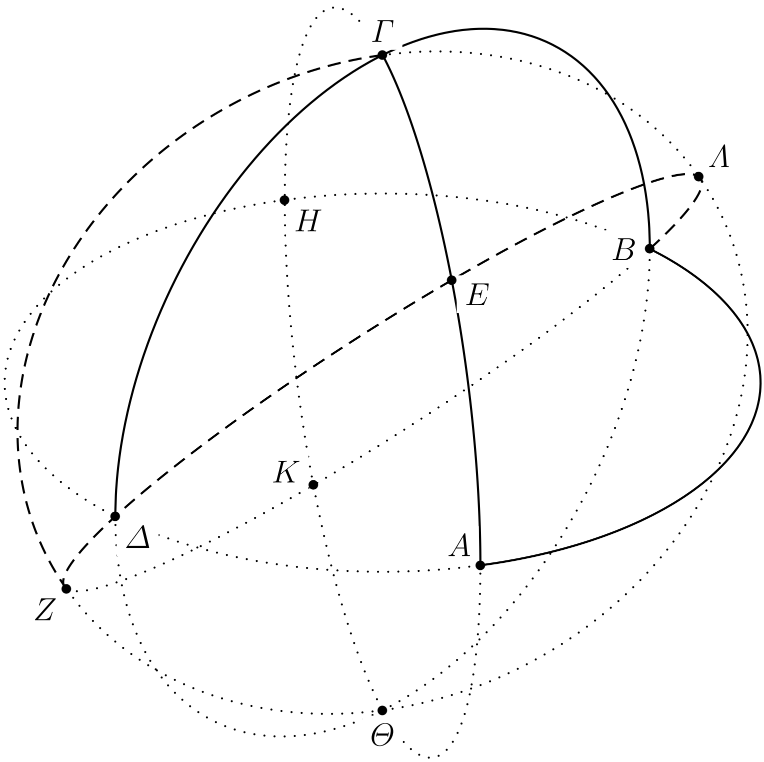

Figure 2: Triangle on a globe

pole, as in Figure 2. Then in triangle $AB\Gamma$, the angles at A and B are right, while the interior angle at Γ may be greater than that, in which case the exterior angle is less.

Thus Euclid's Proposition I.16 fails on the globe. But then so does Postulate 1, in the sense that there is not always a unique "straight line" joining two points. There is no *one* such line, but there are infinitely many, if the points are antipodal.

However, as we have redefined "straight line" on the globe, so we may redefine "point" to mean pair of antipodal points,

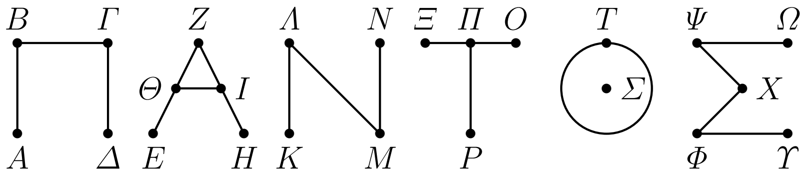

Figure 3: Theorem as diagram

such as, in Figure 2, $\{A, H\}$, $\{B, \Delta\}$, $\{\Gamma, \Theta\}$, $\{E, K\}$, or $\{Z, \Lambda\}$. This way, any two "straight lines" meet at exactly one "point."

With our new conception of straight lines and points, perhaps we still violate Postulate 2, in the sense that we cannot extend a "straight line" if it is already a full great circle. However, it is not clear that Euclid intended the postulate to exclude spheres. He did do research in spherical geometry [15, pp. 348–9].

Euclid's mathematics would seem to lack the kind of precision that we need for Gödel's theorem. On the other hand, we may note how, as something written down on a surface, geometry is subject to itself. The proposition that we have been considering, Euclid's I.16, begins with an absolute phrase (a genitive absolute in Greek),

> Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης,
>
> One of the sides of any triangle having been extended.

Instead of assigning a Gödel number to the proposition, we might consider itself as constituting a diagram, the beginning of which is in Figure 3. For the diagram, I have used not minuscule letters as in $\pi\alpha\nu\tau\acute{o}\varsigma$, but capitals as in ΠΑΝΤΟΣ. We can consider these capitals as consisting of the straight lines and a circle determined by certain points—twenty-four points, as it happens, one for each letter of the Greek alphabet.

We saw in the previous section that for arithmetical systems, there is an alternative: be too weak to formulate $\phi(n)$, or be incomplete. We have a corresponding alternative for geometrical systems.

1. If being a geometrical theorem is a geometrical property of the diagram that corresponds to it as in Figure 3, then we can give a geometrical formulation and solution of the Gödel equation (2). In this case, there are true geometrical statements that are not theorems, and thus geometry is incomplete.
2. In the other case, while geometry may prove theorems, it does not prove *that* they are theorems.

I am speaking loosely about geometry. There are various ways of defining systems that can be called geometrical. The point is that, as with arithmetical systems, so with geometrical: none can answer all questions about its subject-matter.

Perhaps nobody ever thought it would. But some people may think physics does, or can, or will. A possible example is the person who says,

> our conscious states clearly depend on the states of our brains somehow. Since our brains are physical objects and physics is quantum mechanical, I suppose quantum theory must come into it.

That's Tim Maudlin, in a 2018 interview with John Horgan [18], where Maudlin expresses the belief

> that the fundamental physical law—when presented

> in the right mathematical language—will be so compellingly simple that we would think that any other structure would be unnecessarily complicated.

If consciousness is something physical, then I would expect it to be governed by the fundamental physical law, if there is one. In that case, when we are conscious of a mathematical question, we ought to be able to answer it by means of the fundamental law; but this possibility conflicts with Gödel.

So it would seem to me, at least. Maudlin seems not to agree, at least when Horgan asks him,

> Does Gödel's incompleteness theorem have implications beyond mathematics? Is it a worm in the apple of rationality?

Maudlin responds,

> No. Absolutely no one should have ever been surprised that mathematical truth cannot be equated with theoremhood in some finite axiomatic system ... All Gödel did was find a clever way to construct a provably unprovable mathematical fact, given any consistent and finite set of axioms to work with. The work is clever but in no way profound.

On the contrary, Gödel's theorem concerns (1) not only *finite* sets of axioms, but also infinite sets, provided we have a rule for writing them down; (2) not *any* such sets, but those that are satisfied by the counting numbers.

Perhaps one should not make grandiose claims about Gödel's work *anyway.* Surely one should not, without knowing something of the mathematical details, which are exemplified in the next section.

6 Algebra

We now work out a precise example of a complete theory. This reflects strength in one sense, weakness in another. As we have suggested a few times, the Incompleteness Theorem is that some systems are too strong to be complete. The system of our theory will not be too strong, because the axioms of the theory itself *will* be strong enough to settle every question in their language.

If you know what the terms mean, the theory will be that of infinite groups that have exponent two. The axioms are

$$\left.\begin{aligned} \forall x\ \forall y\ \forall z\ x(yz) = (xy)z, \\ \forall x\ x\mathrm{e} = x, \\ \forall x\ x^2 = \mathrm{e}, \end{aligned}\right\} \tag{4}$$

along with, for each counting number n,

$$\exists x_1 \ \cdots \exists x_{n+1} \bigwedge_{1 \leqslant j < k \leqslant n+1} x_j \neq x_k. \tag{5}$$

These are written in a language with such a symbol as comes between x and y in $x \times y$ or $x \cdot y$, but is usually suppressed, as in xy; and x^2 means xx. The axioms (4) do not yield a complete theory; for these we need the axioms (5), of which there are infinitely many.

For examples of algebraic proofs, we shall show first that groups in general can be axiomatized as structures where there is an associative binary operation that has a left identity, and every element has a left inverse; and there are groups that are not commutative.

One learns in school how to add and multiply *integers,* or *whole numbers,* which can be positive, negative, or zero. They are conceived as forming a list

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots,$$

infinite in both directions. These numbers compose the set called $\mathbb{Z}$, the letter being memorable as standing for the German Zahl "number."

Let us now create something new: a set to be called M, comprising all of the lists of four integers. We could write such a list as

$$(a, b, c, d),$$

but instead, for convenience, we shall write it as a two-by-two matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using addition and multiplication of integers, we define multiplication of elements of M by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}.$$

Multiplication in $\mathbb{Z}$ is *associative,* which means

$$x(yz) = (xy)z$$

whenever x, y, and z are integers. The same then is true whenever x, y, and z are elements of M (which means each of x, y, and z has four entries, which are integers). In short, both $\mathbb{Z}$ and M satisfy the following axiom:

$$\forall x\ \forall y\ \forall z\ x(yz) = (xy)z.$$

You can read the symbol $\forall$ as "for all …" or "for every …"

Not the symbolism, but the idea appeared in § 3. The symbol $\forall$ itself, or else a combination such as $\forall x$, is called a *universal quantifier.* Łukasiewicz and Tarski say in a footnote [13, p. 600, n. 19],

> The expression 'quantifier' occurs in the work of Peirce ... although with a somewhat different meaning.

A challenge of reading logic is that, in its study of symbolism, it *uses* symbolism, whose precise meaning is important to get straight, although the symbolism and its meaning will vary from author to author. That is true in mathematics generally, but to a less extent. Perhaps the same is true for programming.

Before the upside-down A, other symbols were used to mean "for all ...." Gödel uses $\mathit{\Pi}$, citing Łukasiewicz and Tarski, who in turn cite Peirce [19, p. 54, n. 2]. More precisely, for our $\forall x$, in the printed texts, Gödel uses $x\mathit{\Pi}$, with slanted letter Pi *after* the variable, although Łukasiewicz and Tarski use $\prod x$, with an enlarged upright Pi that extends below the line and is placed before the variable.

Multiplication on $\mathbb{Z}$ is *commutative,* meaning it satisfies the axiom

$$\forall x\ \forall y\ xy = yx.$$

Unlike associativity, commutativity does not pass to M, since for example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

while in the other order the product is different, at least when b or c is not 0:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

Nonetheless, in both $\mathbb{Z}$ and M, there is a multiplicative *identity,* namely an element e satisfying the axiom

$$\forall x\ (\mathrm{e} \cdot x = x \wedge x \cdot \mathrm{e} = x),$$

where $\wedge$ is to be read as "and." Specifically, e is 1 in $\mathbb{Z}$ and I in M, where

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The name for a set equipped with an associative operation and an identity is *monoid.* I don't know when the name was invented.

Now we can observe that $\mathbb{Z}$ is a monoid with respect to addition as well as multiplication. A more precise way to say this is that each of the structures $(\mathbb{Z}, \cdot, 1)$ and $(\mathbb{Z}, +, 0)$ is a monoid. So is $(M, \cdot, \mathrm{I})$.

One can make M into an additive monoid too, but our concern is with the multiplicative structure.

In $\mathbb{Z}$ as an additive monoid, each element a has an *inverse,* namely $-a$. Combining an element with its inverse, in either order, yields the identity. Written multiplicatively, the axiom being satisfied here is

$$\forall x\ (x^{-1} \cdot x = \mathrm{e} \wedge x \cdot x^{-1} = \mathrm{e}).$$

Neither of $(\mathbb{Z}, \cdot, 1)$ and $(M, \cdot, \mathrm{I})$ has an operation of inversion that satisfies this axiom. However, note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix},$$

and likewise

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix},$$

the same thing. If we define

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

then you can check that

$$\det(AB) = \det A \det B$$

for all A and B in M. If we let S consist of the elements A of M such that $\det A$ is 1, then $(S, \cdot, \mathrm{I})$ is a monoid, and it has inverses. The same is true for G, consisting of the elements A of M such that $\det A$ is 1 or -1.

Any monoid that also has inverses is called a *group.* Thus each of

$$(\mathbb{Z}, +, 0, -), \qquad (S, \cdot, \mathrm{I}, {}^{-1}), \qquad (G, \cdot, \mathrm{I}, {}^{-1})$$

is a group. The subset $\{-1, 1\}$ of $\mathbb{Z}$ is also a group with respect to multiplication.

The identity of a monoid is "two-sided," as are inverses in a group. An associative operation may have only a one-sided identity. For example, on any set, if we define an operation $\otimes$ by the rule

$$x \otimes y = y,$$

this means every element is a left identity; but none is a right identity, if there are at least two elements.

Since identities in a monoid are required to be two-sided, there can be only one of them. Indeed, if e and e′ are identities, then

$$\mathrm{e} = \mathrm{e}\mathrm{e}' = \mathrm{e}'.$$

If an element a of a monoid has both a left inverse, a^{-1}, and a right inverse, a', then these are equal, since

$$a^{-1} = a^{-1} \cdot \mathrm{e} = a^{-1} \cdot (a \cdot a') = (a^{-1} \cdot a) \cdot a' = \mathrm{e} \cdot a' = a'.$$

In this case, a has only one left inverse, and one right inverse, precisely because they are the same.

If a set has an associative operation with a *left* identity, and each element has a *left* inverse, then the identity and the inverses must be two-sided, and thus the structure is a group. That is, on the basis of the three axioms

$$\left.\begin{array}{c} \forall x\ \forall y\ \forall z\ x(yz) = (xy)z, \\ \forall x\ \mathrm{e} \cdot x = x, \\ \forall x\ x^{-1} \cdot x = \mathrm{e}, \end{array}\right\} \tag{6}$$

we can prove

$$\begin{gathered} \forall x\ x \cdot \mathrm{e} = x, \\ \forall x\ x \cdot x^{-1} = \mathrm{e}. \end{gathered}$$

Indeed, for any a we have from the axioms

$$\begin{aligned} (a \cdot a^{-1})(a \cdot a^{-1}) &= a \cdot \big(a^{-1} \cdot (a \cdot a^{-1})\big) \\ &= a \cdot \big((a^{-1} \cdot a) \cdot a^{-1}\big) \\ &= a \cdot (\mathrm{e} \cdot a^{-1}) \\ &= a \cdot a^{-1}, \end{aligned}$$

and consequently

$$\begin{aligned} \mathrm{e} &= (a \cdot a^{-1})^{-1} \cdot (a \cdot a^{-1}) \\ &= (a \cdot a^{-1})^{-1} \cdot \big((a \cdot a^{-1})(a \cdot a^{-1})\big) \\ &= \big((a \cdot a^{-1})^{-1} \cdot (a \cdot a^{-1})\big)(a \cdot a^{-1}) \\ &= \mathrm{e} \cdot (a \cdot a^{-1}) \\ &= a \cdot a^{-1}. \end{aligned}$$

Thus left inverses are right inverses. Therefore

$$a \cdot \mathrm{e} = a \cdot (a^{-1} \cdot a) = (a \cdot a^{-1}) \cdot a = \mathrm{e} \cdot a = a,$$

so the left identity is also a right identity.

We have now proved that the theory of groups has the three axioms (6). The theory is not *complete,* since it entails neither the axiom of commutativity nor its negation. Indeed, as we have seen, in some groups, the associative operation is commutative; in some, not. The former groups are called *abelian,* for historical and practical reasons.

The group $\{1, -1\}$ mentioned above is abelian. Moreover, each element is its own inverse; that is, the group satisfies

$$\forall x \; x^{-1} = x.$$

For a group, satisfying that axiom is the same as satisfying

$$\forall x \; x^2 = \mathrm{e},$$

where x^2 means xx. Every group that satisfies one of these axioms is abelian, since for all elements a and b of such a group,

$$ba = ba \cdot \mathrm{e} = ba(ab)(ab) = b(aa)b(ab) = (bb)(ab) = ab.$$

We now know that, to obtain the theory of groups in which each element is its own inverse, we do not need symbols for inverses, but can use the axioms (4). These are true, by definition, in groups that have *exponent two.*

There are axioms that, together, are true precisely in *infinite* structures:

$$\exists x \; \exists y \; x \neq y,$$
$$\exists x \; \exists y \; \exists z \; (x \neq y \land x \neq z \land y \neq z),$$

and so on. We can write the nth statement on this list as in (5). You read $\exists$ as "there exists ... such that," or "for some

…" Again, not the symbolism, but the idea appeared in § 3. The backwards E used to be written as $\sum$, and it or a combination such as $\exists x$ is an *existential quantifier* [19, p. 55]. Thus our theory has, for each counting number n, an axiom saying that at least $n+1$ elements exist.

The theory of infinite groups that have exponent two is complete. That assertion is a theorem, not *of* the theory, but *about* the theory. One proof of the theorem is by *quantifier elimination.* The idea is that, in each of the groups, each system of equations and inequations can be simplified as needed so that the question of whether there is a solution is easily answered. For example, in any of the groups, whatever a, b, c, and d are, the system

$$x \neq a \land x \neq b \land x \neq c \land x \neq d$$

is soluble, simply because the group is infinite. Thus the statement

$$\exists x\ (x \neq a \land x \neq b \land x \neq c \land x \neq d)$$

is equivalent to the quantifier-free statement

$$\mathrm{e} = \mathrm{e}.$$

Since this is (obviously) true, we have proved the statement

$$\forall y\ \forall z\ \forall t\ \forall u\ \exists x\ (x \neq y \land x \neq z \land x \neq t \land x \neq u).$$

Given any equation or inequation, by using associativity and commutativity, along with the axiom that everything is its own inverse, we can always

- move any letter to the other side,
- delete both occurrences of a repeated letter (on the same side or different sides).

For example, given the system

$$xabxca = bxa \wedge xb \neq axcxcxax,$$

we can rearrange the strings of letters to get

$$aabcxx = abx \wedge bx \neq aaccxxxx,$$

then do some deleting, to obtain

$$bc = abx \wedge bx \neq xx.$$

More deleting yields

$$c = ax \wedge b \neq x,$$

and moving the remaining letters around gives

$$x = ac \wedge x \neq b.$$

We can now eliminate the second instance of x, obtaining

$$x = ac \wedge ac \neq b.$$

Whether this is soluble depends simply on whether ac is different from b. Thus the statement

$$\exists x\ (xabxca = bxa \wedge xb \neq axcxcxax)$$

is equivalent to the quantifier-free statement

$$ac \neq b.$$

This statement is false if (as may be) b is just ac. Thus we have disproved the statement

$$\forall y\ \forall z\ \forall t\ \exists x\ (xyzxty = zxy \wedge xz \neq yxtxtxyx).$$

In such a way, we can prove or disprove every statement from our axioms. These therefore give us a complete theory—after we check one more thing.

Might our theory be inconsistent, so that the same statement can be both proved and disproved? No, because there *are* infinite groups in which every element is its own inverse; in other words, the theory of such groups has at least one *model.* One example of such a model consists of the infinite sequences

$$(a_1, a_2, a_3, \dots),$$

where each of the entries a_i belongs to the group $\{1, -1\}$. Thus the elements of the groups are sequences

$$(\pm 1, \pm 1, \pm 1, \dots).$$

We take the product of any two of these by multiplying entry by entry: for example,

$$(1, 1, -1, -1, \dots) \cdot (1, -1, 1, -1, \dots) = (1, -1, -1, 1, \dots).$$

The group has a subgroup, in each element of which, only finitely many entries are -1; this subgroup is a model of the same theory.

7 Logic

In the previous section, we sketched a proof of a completeness theorem: that, in a certain mathematical language, there are certain axioms such that, for every statement of the language, we can prove either that statement or its negation (but not both) from the axioms. The language has symbols for a binary operation (usually called multiplication) and an identity for that operation; the axioms are of infinite groups of exponent two. The collection of all statements that can be proved from those axioms is the *theory* of infinite groups of exponent two. The elements of that theory are *theorems* of infinite groups of exponent two; an example is the theorem that all such groups are abelian. The completeness theorem that we have proved (or again, *sketched* a proof of) is a *logical* theorem.

As geometry comes from surveying, so algebra comes from counting and reckoning. Logic comes from reasoning, especially reasoning about mathematics; but it ends up creating new mathematical objects. A logical theorem is *like* a mathematical theorem. Perhaps it even *is* a mathematical theorem; but then, more precisely, it is a mathematical theorem, *before* being turned into a mathematical object in the manner that Gödel takes advantage of.

We are now going to describe the completeness of certain logical systems, in both *propositional* and *first-order* logic.

As algebra is concerned with equations, so logic is concerned with *formulas.* An equation is a formula, but then so is a system of equations and inequations, and so is the result of appending one or more quantifiers, as in § 6. We talked about "expressions" in § 4, starting with $\phi(n)$; these were formulas. From the beginning, we have talked about "statements"; these are formulas in which all variables have been quantified.

More precisely, we have just described *first-order* formulas. In the logic called propositional (or sentential), the formulas are simpler. We can define them as follows.

1. Each of the variables P, Q, R, and so on is a formula.
2. The constant 0 is a formula.
3. If each of F and G is a formula, then so is $(F \Rightarrow G)$.

Instead of P, Q, R, "and so on," the variables could be given as $P_1, P_2, P_3, \dots$; or as $P, P', P'', \dots$; we just want there to be an infinite list of them. This allows us to define an infinite list consisting precisely of our (propositional) formulas.

The sign $\Rightarrow$ is a *connective.* For each formula that is not just a single variable or constant, there are formulas F and G such that the original formula is $(F \Rightarrow G)$. That these F and G are *uniquely* determined by the original formula is a theorem, albeit one that is often forgotten about. It would be false if we had left off the parentheses in defining formulas as above. The theorem would be true if we always wrote the formula $(F \Rightarrow G)$ with one parenthesis, as

$$(F \Rightarrow G$$

—or with no parentheses, but in a different order, as

$$\Rightarrow F\,G.$$

This would be so-called Polish notation, apparently due to Łukasiewicz; it is convenient for computers, but perhaps not for human readers, and we shall not use it further.

In the formula $(F \Rightarrow G)$, the sign $\Rightarrow$ displayed between F and G is the *principal connective* of the formula. The theorem that every compound formula has a principal connective, which is therefore *the* principal connective of the formula, allows us to compute, for every formula, the value 0 or 1, once we assign such a value to each variable that occurs in the formula. The computation follows the rule given by the following *truth table*:

$(F$	$\Rightarrow$	$G)$
0	1	0
1	0	0
0	1	1
1	1	1

The rule for such tables is that when variables are given the values written below them, then compound formulas take the values written below their principle connectives. Thus, according to the first line (below the top) of the table above, when each of F and G is given the value 0, then $(F \Rightarrow G)$ itself has the value 1.

A formula that always takes the value 1 is a *tautology.* For example, $(F \Rightarrow F)$ is always a tautology, because its truth table is computed as follows.

$(F$	$\Rightarrow$	$F)$
0	1	0
1	1	1

A truth table can always serve as a proof that a given formula is a tautology. Thus, from the list of all formulas, we have a way to select precisely the ones that are tautologies. This observation itself is a kind of completeness theorem, although it can hardly fail to be true. Moreover, the truth-table method for recognizing tautologies is inefficient, in the sense that, if n variables occur in a formula, then its truth table has 2^n lines.

An alternative method for proving tautologies is to designate

- certain tautologies as *axioms,*
- certain ways of deriving new tautologies from old ones as *rules of inference.*

This gives us a new mathematical system.

By Gödel's Incompleteness Theorem, some mathematical systems are strong enough to be incomplete. By contrast, propositional logic is weak enough that it can have complete systems, albeit in a different sense of "system." We shall describe such a system now; it is based on work of Łukasiewicz, which builds on Frege [6, p. 156].

For our convenience, we simplify the official formulas by using two conventions:

1. Outer parentheses are removed.
2. Internal parentheses are removed, when they can be reinstated by assigning priority to arrows on the right.

Our axioms are now three:

1. $P \Rightarrow Q \Rightarrow P$, or officially $\big(P \Rightarrow (Q \Rightarrow P)\big)$;
2. $(R \Rightarrow Q \Rightarrow P) \Rightarrow (R \Rightarrow Q) \Rightarrow R \Rightarrow P$;
3. $\big((Q \Rightarrow 0) \Rightarrow P \Rightarrow 0\big) \Rightarrow P \Rightarrow Q$.

Our rules of inference are two:

1. *Modus Ponens*: From F and $F \Rightarrow G$, infer G (we mentioned this in § 4).
2. *Substitution*: From any formula, infer the formula that results from substituting a particular formula, the same each time, for every occurrence of a particular variable.

We obtain *theorems* by applying rules of inference, first to axioms, and then to other theorems. A record of such applications is a *proof*. Here is an example.

1. $P \Rightarrow Q \Rightarrow P$ by Axiom 1.
2. $P \Rightarrow (P \Rightarrow P) \Rightarrow P$ by Substitution in step 1.
3. $(R \Rightarrow Q \Rightarrow P) \Rightarrow (R \Rightarrow Q) \Rightarrow R \Rightarrow P$ by Axiom 2.
4. $(P \Rightarrow Q \Rightarrow P) \Rightarrow (P \Rightarrow Q) \Rightarrow P \Rightarrow P$ by Substitution in step 3.
5. $(P \Rightarrow (P \Rightarrow P) \Rightarrow P) \Rightarrow (P \Rightarrow P \Rightarrow P) \Rightarrow P \Rightarrow P$ by Substitution in step 4.
6. $(P \Rightarrow P \Rightarrow P) \Rightarrow P \Rightarrow P$ by Modus Ponens from step 2 and step 5.
7. $P \Rightarrow P \Rightarrow P$ by Substitution in step 1.
8. $P \Rightarrow P$ by Modus Ponens from step 6 and step 7.
9. $F \Rightarrow F$ by Substitution in step 8.

Strictly, the proof consists only of the nine formulas, in the order given. The numbers and the explanations help prove to *us* that the list of formulas *is* a proof. The proof establishes $F \Rightarrow F$ as a theorem.

This formula is also a tautology, as we saw earlier from its truth table. Strictly though, the truth table we wrote for $F \Rightarrow F$ was really for $P \Rightarrow P$. The real truth table for $F \Rightarrow F$ has as many lines as that for F itself, with twice the columns and one more. For example, here is the truth table when F is $(P \Rightarrow Q) \Rightarrow 0$ (the row added to the bottom give the stages in which the columns are filled in):

$((P$	$\Rightarrow$	$Q)$	$\Rightarrow$	$0)$	$\Rightarrow$	$(P$	$\Rightarrow$	$Q)$	$\Rightarrow$	0
0	1	0	0	0	1	0	1	0	0	0
1	0	0	1	0	1	1	0	0	1	0
0	1	1	0	0	1	0	1	1	0	0
1	1	1	0	0	1	1	1	1	0	0
2	3	2	4	1	5	2	3	2	4	1

We need not actually do all the work of writing out such a truth table, if we already know the table for $P \Rightarrow P$; but this is precisely because Substitution is a legitimate rule of inference.

By saying Substitution is legitimate, we mean that, from tautologies, it produces only tautologies. The same is true for *Modus Ponens.* Thus every theorem is a tautology. The converse is also true, albeit much harder to prove (and we are not going to try to prove it): every tautology is a theorem. This means our system of axioms and rules of inference is complete. Thus there is a completeness theorem: not a mathematical theorem *of* our system, but a logical theorem *about* the system.

We could define an alternative system in which every tautology is an axiom; this would automatically be complete. Likewise (see § 5), Euclid could have given us full use of a set square from the beginning, or for that matter given us, as postulates, all of his propositions, or at least the "obvious" ones. However, the point is to see how little we can get by with.

In the first-order logic of addition and multiplication, the counting numbers have a complete theory, consisting of the statements that are true of the natural numbers. The problem is, we have no way of making a *complete* list of those statements. Indeed, Gödel's Incompleteness Theorem is that any such list is inevitably incomplete.

This is not like Cantor's theorem (if you know it) that the real numbers cannot be listed. The *statements* of the first-order logic of addition and multiplication can be listed, but there can be no algorithm for picking out which statements are true of the counting numbers. We may be able to figure out whether a particular statement is true; indeed, this is happens in *number theory* (although number-theorists do not restrict themselves to first-order logic).

There *is* an algorithm for picking out the statements that are true, no matter how addition and multiplication are defined: this is one way to formulate Gödel's *Completeness* Theorem. In the more usual formulation, every one of those true statements has a proof (like the one above for $F \Rightarrow F$); but we can also make a list of all proofs. There is no algorithm for picking out the statements that are *not* always true, but proving this needs more than Gödel's theorems.

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