A Criteriology of Mathematics

David Pierce
Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
mat.msgsu.edu.tr/~dpierce/
polytropy.com/about/

November 19, 2021
July 6, 2025

Abstract

By one analysis, ethics and logic are respectively the sciences of practical and theoretical thought. In the terminology of R. G. Collingwood, they are neither normative nor empirical, but *criteriological*: they give an account, neither of what we are to do, nor of what we are already doing, but of what we are *trying* to do, by our own standards. In mathematics, we are trying to prove theorems, albeit sometimes in a broad sense: a solution to any kind of mathematical problem involves a proof – which may be only implicit – that the solution is correct. In theory, we can work alone on proving theorems. In practice, we need others to agree with the proofs. This agreement cannot be coerced. For the same reason, we have to be ready to recognize our own errors, which may be pointed out by others. Thus, briefly, mathematics is both individual and universal. It is a free activity, in which we have the right and the responsibility to share our understanding and resolve disputes peacefully. This is our fundamental ethic. It may be common

with other sciences, while retaining its peculiar features. We do well to keep it in mind when other sorts of disputes arise.

Like any other science, mathematics can be put to good or bad use. We may be able to inhibit the bad use, if we are clear about we are really trying to do in mathematics. I propose that this is to create something that cannot be hoarded or fought over.

Covetousness and even personal ambition are foreign to mathematics as such. I say this, while recognizing such counterarguments as G. H. Hardy's in *A Mathematician's Apology* [11, §§2, 29; pp. 65–6, 144–5]:

> [M]y apology is bound to be to some extent egotistical. I should not think it worth while to apologize for my subject if I regarded myself as one of its failures.
>
> Some egotism of this sort is inevitable, and I do not feel that it really needs justification. Good work is not done by 'humble' men.
>
> . . . .
>
> I cannot remember ever having wanted to be anything but a mathematician . . . I do not remember having felt, as a boy, any *passion* for mathematics . . . I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys, and this seemed to be the way in which I could do so most decisively.

There is evidently competition among people who do mathematics. However, even the competition in a footrace or tennis match is mitigated by an agreement to abide by certain rules, as enforced by certain referees. In mathematics, the players are the only referees. One of us may arrive first at a goal, such as the proof of a theorem; but we can be known to have done so, only by others whom we have brought along, step by step, to that goal.

Advice of the ancient Greek Stoic philosopher Epictetus may be useful here. This is from the *Encheiridion* [7, XIX, p. 242]:

> Take care then when you observe a man honored before others or possessed of great power or highly esteemed for any reason, not to suppose him happy, and be not carried away by the appearance. For if the nature of the good is in our power, neither envy nor jealousy will have a place in us. But you yourself will not wish to be a general or senator (*πρύτανις*) or consul, but a free man: and there is only one way to this, to despise (care not for) the things which are not in our power.

We may well object to such advice, because what is in our power is not always clear. The question remains: Do we desire a particular title, such as professor or Fields medalist? Or do we want freedom, especially freedom from ignorance?

It is easy to give advice, or to promulgate a code of conduct. It is harder to get people to follow it, and this is shown by refusals to take vaccines or to stop digging and drilling for fossil fuel. I think the difficulty here corresponds to a subtle distinction between sciences like medicine and engineering on the one hand, and ethics on the other. The former are normative, and they could have been called “criterial,” because they establish norms or criteria for getting something done, be it reducing a fever or letting a building survive an earthquake. Ethics may deal in criteria like these, but they cannot be imposed on us the way an antipyretic can be imposed on our body. Ethics is not criterial, but criterio-*logical,* because it gives an account, a *logos,* of the criteria that we already use, or try to use, when we are doing our best.

I ask then what kind of person we want to be. Thus, it seems, I take the side of the ancient Greeks, as distinct from the Moderns, by the account of Jonathan Haidt in *The Happiness Hypothesis* [10, p. 163]:

> Where the Greeks focused on the character of a person and asked what kind of person we should each aim to become, modern ethics focuses on actions, asking when a particular action is right or wrong. Philosophers wrestle with life-and-death

> dilemmas: Kill one to save five? Allow aborted fetuses to be used as a source of stem cells? Remove the feeding tube from a woman who has been unconscious for fifteen years? Nonphilosophers wrestle with smaller quandaries: Pay my taxes when others are cheating? Turn in a wallet full of money that appears to belong to a drug dealer? Tell my spouse about a sexual indiscretion?

As mathematicians, we may ask whether we should

- ignore errors in our work;
- pay to publish that work;
- work for an employer whose aims are secret;
- ridicule people whose understanding is poor;
- pre-judge somebody's understanding in the first place.

Answers would seem to depend on what we are trying to do as mathematicians.

A first step is to recognize that we are indeed trying to do something. We are thus *free.* Our freedom involves a certain antithesis, which consists of

- the *right* to demand reasons for the assertions of our colleagues, and
- the *responsibility* to provide such reasons ourselves, and to acknowledge when they are shown to be inadequate.

Freedom in this sense should perhaps be found in any science, even in politics. Mathematics is peculiar through the involvement of a second antithesis. The knowledge or truth that we seek is

- *individual,* because we judge it for ourselves, as we might a work of art, and nobody else can order us to accept it;
- *universal,* because we still must all agree on the same claim, and if we do not agree, we cannot fight, but must work amicably towards a resolution.

We may detect that antithesis in the enigmatic Heraclitus, who is reported to have said [15, **D2**, p. 139],

> But although the account (*logos*) is in common (*xunos*), most people live as though they had their own thought (*phronêsis*).

We can translate *logos* also as reason.

A third antithesis arises that may be at play in any scientific pursuit, while still bringing out the peculiarity of mathematics.

- *Competition* is possible, as Hardy recalled from his youth.
- *Collaboration* is essential, especially in mathematics, where the only real confirmation of any success comes from thought alone, and

as Heraclitus also said [15, **D29**, p. 153],

Thinking (*phroneein*) is in common for all.

The remainder of this essay is an elaboration of the foregoing, in four sections, as follows.

1. **The Individual and the Universal.** Timothy Gowers observes that, with the modern advent of the formal proof, mathematics acquires a method of resolving all of its disputes, at least in principle, and thus becomes unique among the sciences. Even in ancient times, Ptolemy could distinguish our subject by saying, "only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously." Following Leibniz though, Hannah Arendt notes a commonality among "mathematical, scientific, and philosophical truths," which can be "assigned to the common species of rational truth as distinguished from factual truth . . . Facts and events are infinitely more fragile things than axioms, discoveries, theories . . . "

2. **Right and Responsibility.** Mathematics is distinguished from philosophy and from natural science in what a practitioner has a right to demand and a responsibility to supply – or to learn. Philosophy lets you question anything, even a mathemat-

ical assertion – such as the uncountability of the continuum – that makes no intuitive sense, though it has a proof. In mathematics, one has to learn that the proof is enough to establish the assertion, at least in principle. Moreover, the assertion is established by nothing else: not conformity with the natural world or with what one reads in other books. Conversely, outside mathematics, one cannot strictly apply such a rule as that exactly one of P and not-P is true.

3. **Criteriological Science.** The distinguishing and naming of some sciences as criteriological is due to R. G. Collingwood. As mathematicians, we may understand how ethics is criteriological by considering how logic is too. Logic tries to tell us the rules that we ourselves are already trying to follow. It is up to us, individually and collectively, to decide whether those rules have themselves been given correctly. This comes home to me from the experience of publishing a theorem to which a colleague then found a counterexample, after which I found the error in my proof.

4. **Competition and Collaboration.** Hardy moved beyond the stage when "I still thought of mathematics as essentially a 'competitive' subject." What inspired him was Camille Jordan's

Cours d'analyse. I myself was inspired by Michael Spivak's *Calculus* (2nd ed., 1980), where the student is treated as a colleague in a collaborative effort.

1 The Individual and the Universal

Regarding the putative individuality and universality of mathematics, at least two objections are possible:

1. They do not always obtain, even in mathematics.
2. They ought to obtain in any science.

As to the first point, by thinking about what we do and are *trying* to do as mathematicians, we may approach more nearly our ideal, whatever that may turn out to be.

As to the second point, it is ultimately for practitioners of other sciences to work out what their own ideal is. That they do not follow the mathematical ideal would seem to be the argument of Timothy Gowers in *Mathematics: A Very Short Introduction* [9, Ch. 3, "Proof," p. 40],

> . . . the fact that disputes can *in principle* be re-

> solved does make mathematics unique. There is no mathematical equivalent of astronomers who still believe in the steady-state theory of the universe, or of biologists who hold, with great conviction, very different views about how much is explained by natural selection, or of philosophers who disagree fundamentally about the relationship between consciousness and the physical world, or of economists who follow opposing schools of thought such as monetarism and neo-Keynesianism.

Alongside Gowers's examples of contrarian astronomers, biologists, philosophers, and economists, one might place mathematicians who reject the law of the excluded middle. However, to prefer or insist that proofs be constructed within intuitionistic logic is not to dispute whether any particular proof within classical logic is correct by the rules of that logic.

The idea of mathematics as both individual and universal is hinted at in a recent essay on Hannah Arendt by Rebecca Panovka called "Men in Dark Times" [19]:

> Arendt first distinguishes rational truth (mathematical truth, philosophical truth – anything that can be proven by axiom) from factual truth. The difference here is between a statement like

> "2 + 2 = 4" and one like "It rained in Reykjavík yesterday." Whereas rational truth must be the case, Arendt writes, "facts have no conclusive reason whatever for being what they are." Believing them requires a degree of trust in witnesses, historians, and scientists. If rational truth provides the basis for philosophical speculation, then facts perform an analogous role for political thought. When we enter the public sphere to debate a policy or decision, proper participation is only possible if we rely on a shared set of facts.

What is here called factual truth, as distinct from mathematical or more generally rational truth, can hardly be established by the individual working alone. Moreover, any universality of the factual truth will not extend beyond those of us who already "rely on a shared set of facts."

Panovka is talking about the essay "Truth and Politics" [1], where Hannah Arendt herself traces the distinction between rational and factual truth to one of the inventors of calculus:

> Leibniz assigned mathematical, scientific, and philosophical truths to the common species of rational truth as distinguished from factual truth . . . Facts and events are infinitely more fragile things than axioms, discoveries, theories – even the most wildly speculative ones – produced by the human

> mind; they occur in the field of the ever-changing affairs of men . . . Once they are lost, no rational effort will ever bring them back. Perhaps the chances that Euclidean mathematics or Einstein's theory of relativity – let alone Plato's philosophy – would have been reproduced in time if their authors had been prevented from handing them down to posterity are not very good either, yet they are infinitely better than the chances that a fact of importance, forgotten or, more likely, lied away, will one day be rediscovered.

However unlikely it may be, the reproducibility of Euclid or Einstein or Plato would be an aspect of what I am calling universality. Euclid may have the strongest claim to this universality. As for what I am calling individuality, Arendt herself attributes it to all four kinds of truth, mathematical, scientific, philosophical, and factual:

> Statements such as "The three angles of a triangle are equal to two angles of a square," "The earth moves around the sun," "It is better to suffer wrong than to do wrong," "In August 1914 Germany invaded Belgium" are very different in the way they are arrived at, but, once perceived as true and pronounced to be so, they have in common that they are beyond agreement, dispute, opinion, or consent. For those who accept them, they are not changed by the numbers or lack of

> numbers who entertain the same proposition; persuasion or dissuasion is useless, for the content of the statement is not of a persuasive nature but of a coercive one.

The coercion here comes from the truth itself, not from another person:

> What Mercier de la Rivière once remarked about mathematical truth applies to all kinds of truth: "*Euclide est un véritable despote; et les vérités géométriques qu'il nous a transmises, sont des lois véritablement despotiques.*"[1] In much the same vein, Grotius, about a hundred years earlier, had insisted – when he wished to limit the power of the absolute prince – that "even God cannot cause two times two not to make four." He was invoking the compelling force of truth against political power . . .

In the Platonic dialogue called *Gorgias* [21], Socrates works out an argument that, as Arendt mentions, it is better, or at least less bad, to suffer than to do injustice; but even if the interlocutors Polus and Callicles accept this rationally, they may not act on it. This is the general problem of ethical codes mentioned earlier, that they are not

[1]Euclid is a veritable despot; and the geometrical verities that he has transmitted to us are truly despotic laws.

self-enforcing.

In Book I of Plato's *Republic* [22], while the characters called Cephalus, Polemarchus, Thrasymachus, Glaucon, and Adeimantus all accept tacitly that there is one thing called justice, they share with Socrates no common understanding of what it is.

We may likewise accept that there is one thing called the universe, although the theories of it are many. Arendt mentions a putative truth about it: "The earth moves around the sun." That's the Copernican, heliocentric theory, as distinct from the Ptolemaic, geocentric theory. By the Einsteinian theory, nothing moves around anything else in any absolute sense. Perhaps anybody who studies the physical universe must inevitably end up with Einstein's theory of relativity. Arendt suggests that this is more likely than their arriving at Plato's philosophy, but I'm not sure. In any case, Euclid's ancient theory of geometry is still somehow our theory too.

We likewise retain, as our own, Ptolemy's theory of geometry, even explicitly in what we call Ptolemy's Theorem [23, I.10, H36–7, pp. 50–1]. The theorem relates the sides and diagonals of a cyclic quadrilateral, and Ptolemy uses it, in a way that we can recognize as correct, to compute values for a table of sines.

The analysis, discussed by Hannah Arendt, of "the common species of rational truth" into "mathematical, scientific, and philosophical truths" is, for Ptolemy, the analysis of theoretical philosophy into mathematics, physics, and theology. Here he is on the distinctiveness of mathematics [23, I.1, H5, p. 35]:

> Aristotle divides theoretical philosophy too, very fittingly, into three primary categories, physics, mathematics, and theology . . .
>
> From all this we concluded: that the first two divisions [namely theology and physics] of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry.

Ptolemy too sees the possibility of a mathematical contribution to ethics [23, I.1, H6, p. 36–7]:

> With regard to virtuous conduct in practical ac-

> tions and character, this science, above all things, could make men see clearly; from the constancy, order, symmetry and calm which are associated with the divine, it makes its followers lovers of this divine beauty, accustoming them and reforming their natures, as it were, to a similar spiritual state.

If only it were so! A precept, once stated, may be perverted. Having the title of a mathematician does not make you virtuous, any more than being a head of state would make it impossible to break the law. If you have a title, you still have to decide what it means to you.

2 Right and Responsibility

I said that in mathematics we had the right to demand, and the responsibility to supply, reasons for our assertions. This *right,* at least, may be less broad in mathematics than in philosophy, by the account of Wilfrid Hodges in "An editor recalls some hopeless papers" [13]. The hopeless papers are attempts to refute Cantor's diagonal proof that the real numbers exceed the natural numbers in cardinality. By Hodges's guess (and it is explicitly only that),

> the problem with Cantor's argument is as follows. This argument is often the first mathematical argument that people meet in which the conclusion bears no relation to anything in their practical experience or their visual imagination.

This may explain my own response, when I encountered Cantor's argument in popular books that I read in high school, such as Lillian R. Lieber's *Infinity* [17]. Cantor's conclusion did not make sense, and therefore I thought the argument had to be wrong.

My *feel* for the mathematics must have been insufficiently developed, by the account of Brandon Larvor in "Feeling the force of argument" [16]:

> Higher education requires students to make judgments about the evidence and arguments placed before them, and all judgment has an aesthetic aspect. A mathematics student must be *struck* by the validity and elegance of a proof; a science student must *feel* the weight of evidence (or the lack of it). In the humanities, a lot of bad writing is the result of students trying to articulate and defend judgments that they have copied from secondary sources but have not felt in their viscera. This is not to say that judgment is all unreasoned, inarticulate conviction. Nor is it to suggest that logical relations between premises and conclusions are

> somehow subjective. On the contrary, the point is that students should perceive logical relations as objective realities.

Larvor's statement about "the humanities" would seem to hold, *mutatis mutandis,* for mathematics, where students are also asked "to articulate and defend judgments" – about propositions given to them to be proved. As for the skeptics submitting papers to Wilfrid Hodges,

> Several of the authors said that they had trained as philosophers, and I suspect that in fact most of them had. In English-speaking philosophy (and much European philosophy too) you are taught not to take anything on trust, particularly if it seems obvious and undeniable. You are also taught to criticise anything said by earlier philosophers. Mathematics is not like that; one has to accept some facts as given and not up for argument.

"One has to accept some facts" – one has the *responsibility* of accepting some facts, and one has to develop a feel for this responsibility.

The present essay being more philosophy than mathematics, each of its statements P is open to criticism. When P is "Mathematics is universal," some persons assert $\neg P$. They may cite such works as Alan J. Bishop, "Western mathematics: the secret weapon of cultural imperialism" [3]. Here then is a dispute that needs resolving – not, however, by the victory of one of P and $\neg P$ over the other, but rather by the clarification and refinement of each of these propositions. I like how R. G. Collingwood puts it in *An Essay on Philosophical Method* [6, pp. 105–6]:

> The normal and natural way of replying to a philosophical statement from which we dissent is by saying, not simply 'this view seems to me wrong', but 'the truth, I would suggest, is something more like this', and then we should attempt to state a view of our own. This view certainly need not be on the tip of our tongue; it may be something with which our mind, as Socrates would say, is pregnant, and which needs both skill and pains to bring it to birth . . .
>
> This is not a mere opinion. It is a corollary of the Socratic principle . . . that there is in philosophy no such thing as a transition from sheer ignorance to sheer knowledge, but only a progress in which we come to know better what in some sense we know already.

In mathematics we do make such transitions, from not knowing which of P and $\neg P$, to knowing one of them; but not everything is mathematics.

When Hodges says that in our subject, "one has to accept some facts as given and not up for argument," I suppose definitions are examples. In teaching set theory, I define the intersection of a set or class as the class of all such sets as belong to each set in the original set or class. This would seem to be a standard definition, but it paradoxically makes the intersection of the empty set the universal class, which is not a set. Some books require the intersection of a set to be a set. This means the intersection of the empty set must be either left undefined or defined arbitrarily, perhaps as the empty set itself. When I marked as wrong a student's assertion

$$\bigcap \varnothing = \varnothing,$$

he showed me a reputable mathematical source that made precisely this definition. The experience has stayed with me as an example of how

- mathematics is different from natural science;
- students may not see this.

Different natural scientists may use different definitions too; I recall, for example, several definitions of acid and base from high-school chemistry. However, the ultimate concern in natural science is not logical deductions from definitions and axioms.

The subject of *The Autobiography of Malcolm X* [18, p. 29] recalls living in a detention home in Mason, Michigan, and attending seventh grade at Mason Junior High School:

> I'm sorry to say that the subject I most disliked was mathematics. I have thought about it. I think the reason was that mathematics leaves no room for argument. if you made a mistake, that was all there was to it.

I have heard such an objection from others, too. I think it is an objection to the coerciveness of truth that Hannah Arendt talked about. This coerciveness has a positive side: when you are right, nobody can argue the rightness away. This may be what draws many of us *to* mathematics.

3 Criteriological Science

In any science, one has to accept *something* as not being up for argument. Each science will have its ethic, in the sense of an accepted way of doing things.

The word science by itself often stands for natural or physical or empirical science. I take science to be the systematic pursuit of knowledge of any kind.

Science then is a form of thought, even thought in the highest sense.

Science can have many objects. Even thought itself can be an object, as it is for ethics and logic, whose founding is described by Collingwood in *An Essay on Metaphysics* [5, Ch. X, p. 108]:

> The science of mind . . . must describe the self-judging function which is part and parcel of all thinking and try to discover the criteria upon which its judgements are based.
>
> This demand was recognized by the Greeks; and in their attempts at a science of thought they tried to satisfy it. They constructed a science of theoretical thought called logic and a science of practical thought called ethics.

Etymologically speaking, theory is about seeing things, as in a theater; the words "theory," "theater," and for that matter "theorem" all come from the Greek θέα, meaning sight or spectacle. Practice is about solving problems, although it seems to be only accidental that the words "practice" and "problem" start with the same sounds. Practical and theoretical *sciences* can be roughly distinguished:

- Engineering and medicine are practical, aimed at getting things done.

- Physics and chemistry are (more or less) theoretical, aimed at seeing how things are.

These sciences are instances of sustained thinking; however, they are not themselves sciences *of* thought.

Thought in the strictest sense – not daydreaming, but the active search for a cure for a disease, or the proof of a theorem, or the best response to an insult – such thought is *successful* or not (or somewhere in between). A science of thought has to account for the possibility of success or failure in the thought that it studies.

Thus for example logic formulates rules that our proofs must respect. Such a rule might also be called

- a norm, from the Latin word *norma* for a carpenter's square;
- a criterion, from the Greek *κριτήριον*, ultimately from the verb *κρίνω*, meaning *to separate, choose, judge.*

Logic might therefore be called a normative science, or perhaps a "criterial" one. A better term is *criteriological,* because the science gives an explicit λόγος, or account, of the criteria that we already use implicitly in assessing our proofs. Once those criteria become explicit, we can write out so-called “formal” proofs, which in principle anybody can check, step by step, mechanically. The proof may or may not avoid using the law of the excluded middle, mentioned earlier. In any case, a computer can check the proof. This doesn’t mean there is a physical, empirical test for the correctness of a proof. Any particular physical machine, including a brain, may fail to be in proper working order, according to *our* judgment.

Collingwood introduced the term *criteriological* in a note at the end of Chapter VIII, “Thinking and Feeling,” of *The Principles of Art* [4]. He elaborated in later works, such as *An Essay on Metaphysics,* already mentioned, where Chapter X is called “Psychology as the Science of Feeling”; Chapter XI, “Psychology as the Pseudo-Science of Thought.” While it may be desirable to have some feelings and avoid others, feeling itself is neither successful nor unsuccessful. It just happens, and therefore it can be studied empirically, as it is by the science called psychology. By contrast, being the *expression* of feeling, *art* can be successful or not, by its own standards; this makes aesthetics criteriological.

It is desirable to have the feeling of proving a theorem. However, I think what we really want is to prove it *correctly.* No scrutiny of our physical or emotional state can tell us when we have achieved this. All we can do is think through our proof again and ask others to think it through with us. The tools supplied by symbolic logic may be useful in this.

I worked for years to state, prove, and publish a theorem that turned out to be false. I learned that it was false when a colleague sent me a counterexample; then I found an error in my proof. I went back to the drawing board and found the correct theorem. I *say* it is correct. It passed through peer review and was published, though the same had been true for the earlier theorem. The new theorem has been cited and used by others. However, if you *really* want to know whether it is correct, you have no choice but to work through the proof, or perhaps find your own – or a counterexample, if the putative theorem is still not really that. No brain scan or other physiological study will help you in this work. Another mathematician may help.

As a science of thought alongside logic, ethics too is criteriological. I cannot tell you what you ought to do; I can only invite you to consider what your own standards are, while I try to understand my own.

Physics is *not* criteriological, because its objects are not trying to be anything other than they are. For Ptolemy, perhaps air was striving to go up, stones to go down, and heavenly bodies to move in circles; but all this meant practically was that Ptolemy analyzed the orbits of those heavenly bodies as sums of uniform circular motions.

Today we still describe those orbits as the result of striving. Following Newton, we call that striving *gravity,* or rather we identify it with the gravity already recognized on earth.

The distinction between the criteriological and the empirical is implicit in a comment by Richard Feynman in "Cargo Cult Science" [8, pp. 342–3]:

> But this long history of learning how to not fool ourselves – of having utter scientific integrity – is, I'm sorry to say, something that we haven't specifically included in any particular course that I know of. We just hope you've caught on by osmosis.

If the history and ethics of physics are indeed not taught in physics courses, this is because they are not strictly part of physics. Physics is a natural science; ethics and history are criteriological sciences. However, every physicist, even every scientist, including the mathematician, must be ethicist enough to understand Feynman's advice [8, pp. 343]:

> The first principle is that you must not fool yourself – and you are the easiest person to fool. So you have to be very careful about that. After you've not fooled yourself, it's easy not to fool other scientists. You just have to be honest in a conventional way after that.
>
> I would like to add something that's not essential to the science, but something I kind of believe, which is that you should not fool the layman when you're talking as a scientist.

Is it really not essential that the scientist not fool the laity?

4 Competition and Collaboration

Hardy would seem to *encourage* fooling people, at least in moderation, since otherwise oneself becomes the fool [11, §2, p. 66]:

> It is one of the first duties of a professor, for example, in any subject, to exaggerate a little both the importance of his [*sic*] subject and his own importance in it. A man who is always asking 'Is what I do worth while?' and 'Am I the right per-

> son to do it?' will always be ineffective himself and a discouragement to others.

Hardy does warn against going too far in the sequel:

> [The professor] must shut his eyes a little and think a little more of his subject and himself than they deserve. This is not too difficult: it is harder not to make his subject and himself ridiculous by shutting his eyes too tightly.

Our concern now is not simply what mathematicians do, but what we *ought* to do, as mathematicians. What we ought to do includes what we *have* to do, and this includes being clear about what we *are* doing or are trying to do.

We never really know what we can do until we actually do it. *Trying* to do it requires a self-confidence that, strictly speaking, is justified only in retrospect. If we have that confidence anyway, it may spill out in the boastfulness that Hardy recommends.

More likely, this boastfulness is an attempt to fight back one's own self-doubt. Such doubt is a general problem, which may be addressed in advice columns. Carolyn Hax was asked:

> How does a person raise their self-esteem? Telling myself multiple times a day "You are pretty!" and

“You are smart!” hasn’t worked.

Hax reported this question in “How to improve self-esteem when meds and mantras aren’t working?” [12]. How would *we* answer such a question? Following Hardy, would we suggest, “Act like a big shot (though not too much)”?

Hardy has the confession quoted earlier: “I wanted to beat other boys, and [mathematics] seemed to be the way in which I could do so most decisively.” You can beat somebody with a stick; but if you want to beat them at mathematics, they have to agree with you when it happens.

That is true for tennis also: beating the competition at Wimbledon requires playing by the rules, which everybody agrees on.

There is a difference, like that between normative and criteriological science. In a dispute at Wimbledon, though they may grumble about it, players have to defer to the authority of the referees. In mathematics, each of us is an authority. Disputes are resolved, certainly not by shouting, but not by deference either. We somehow have to share our authority. This is the universality of mathematics. This or that person may prove a theorem first, but the priority has no meaning unless the proof has been shared with all of us. The only way to know that somebody else has a proof is to have learned the proof for ourselves.

Hardy learned more about what mathematics was [11, §29, pp. 146–7]:

> I found at once, when I came to Cambridge, that a Fellowship implied 'original work', but it was a long time before I formed any definite idea of research . . . I was really quite ignorant, even when I took the Tripos, of the subjects on which I have spent the rest of my life; and I still thought of mathematics as essentially a 'competitive' subject. My eyes were first opened by Professor Love . . . the great debt which I owe to him . . . was his advice to read Jordan's famous *Cours d'analyse*; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant.

I have spend little time with Jordan's *Cours* [14], but my own inspiration was the *Calculus* [24] of Michael Spivak. He showed that our subject was not simply handed down from above, but built up by all of us, collectively and individually. It is somehow thanks to Spivak that I have had the nerve to criticize the logic of his own account of the natural numbers [2, 20].

I suggested that scientists had to decide on their own ideals. It's not quite so simple, since I am also convinced by Collingwood that the ideal of creating a purely empirical science of thought is a contradiction in terms. One may use mathematics as one likes. However, with Hardy, one should probably think about what it is that one *really* likes; for mathematics is a way to create a kind of wealth that only increases when shared.

References

[1] Hannah Arendt. Truth and politics. *The New Yorker*, pages 49–88, 25 February 1967.

[2] John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*, chapter 1, §2, Example 3 (Pierce), pages 37–8. Cambridge University Press, 2018.

[3] Alan J. Bishop. Western mathematics: the secret weapon of cultural imperialism. *Race & Class*, 32(2), 1990.

[4] R. G. Collingwood. *The Principles of Art*. Clarendon Press, Oxford, 1938. Issued as an Oxford University Press paperback 1958.

[5] R. G. Collingwood. *An Essay on Metaphysics*. Clarendon Press, Oxford, revised edition, 1998. With an Introduction and additional material edited by Rex Martin. Published in paperback 2002. First edition 1940.

[6] R. G. Collingwood. *An Essay on Philosophical Method*. Clarendon Press, Oxford, new edition, 2005. With an Introduction and additional material edited by James Connelly and Giuseppina D'Oro. First edition 1933.

[7] Epictetus. *A Selection from the Discourses of Epictetus with the Encheiridion*. Henry Altemus, Philadelphia, n.d. Translated by George Long.

[8] Richard P. Feynman. *"Surely You're Joking, Mr. Feynman!"*. W.W. Norton & Company, New York and London, 1985. Adventures of a Curious Character; as told to Ralph Leighton; edited by Edward Hutchings.

[9] Timothy Gowers. *Mathematics: A Very Short Introduction*. Very Short Introductions. Oxford University Press, Oxford, 2002.

[10] Jonathan Haidt. *The Happiness Hypothesis: Putting Ancient Wisdom and Philosophy to the Test of Modern Science*. Arrow Books, Random House, London, 2006.

[11] G. H. Hardy. *A Mathematician's Apology*. Cambridge University Press, Cambridge, 1982. With foreword by C. P. Snow. Reprint of the 1967 edition. First edition 1940.

[12] Carolyn Hax. How to improve self-esteem when meds and mantras aren't working? *The Washington Post*, 14 July 2021. www.washingtonpost.com/lifestyle/2021/07/14/carolyn-hax-self-esteem-happy-with-myself-comparison/, accessed November 4, 2021.

[13] Wilfrid Hodges. An editor recalls some hopeless papers. *The Bulletin of Symbolic Logic*, 4(1):1–16, March 1998. www.jstor.org/stable/421003.

[14] C. Jordan. *Cours d'Analyse de L'École Polytechnique.* Gauthier-Villars, Paris, 3 edition, 1909.

[15] André Laks and Glenn W. Most, editors. *Early Greek Philosophy III*, volume 526 of *the Loeb Classical Library.* Harvard University Press, Cambridge MA and London, 2016. Early Ionian Thinkers, Part 2.

[16] Brandon Larvor. Feeling the force of argument. In A. Kenkmann, editor, *Teaching Philosophy*, pages 134–152. Continuum, 2009. `uhra.herts.ac.uk/bitstream/handle/2299/5831/903260.pdf`, accessed November 4, 2021.

[17] Lillian R. Lieber. *Infinity: Beyond the Beyond the Beyond.* Paul Dry Books, Philadelphia, 2007. Edited and with a new foreward by Barry Mazur. Illustrations by Hugh Gray Lieber. First edition 1953.

[18] Malcolm X. *The Autobiography of Malcolm X.* Grove Press, New York, 1966.

[19] Rebecca Panovka. Men in dark times. *Harper's*, August 2021.

harpers.org/archive/2021/08/men-in-dark-times-hannah-arendt-post-truth/, accessed November 4, 2021.

[20] David Pierce. Induction and recursion. *The De Morgan Journal*, 2(1):99–125, 2012. demorgangazette.files.wordpress.com/2021/02/f424a-david_pierce.pdf, accessed October 16, 2021.

[21] Plato. *Plato III. Lysis, Symposium, Gorgias*. Number 166 in the Loeb Classical Library. Harvard University Press and William Heinemann, Cambridge MA and London, 1983. With an English Translation by W. R. M. Lamb, M.A.. First printed 1925.

[22] Plato. *The Republic of Plato*. Basic Books, New York, 2016. Translated and with an Interpretive Essay by Allan Bloom. With a New Introduction by Adam Kirsch. First edition 1968.

[23] Ptolemy. *Ptolemy's Almagest*. Princeton University Press, 1998. Translated and annotated by G. J. Toomer. With a foreward by

Owen Gingerich. Originally published by Gerald Duckworth, London, 1984.

[24] Michael Spivak. *Calculus*. Publish or Perish, Berkeley, California, 2nd edition, 1980.