

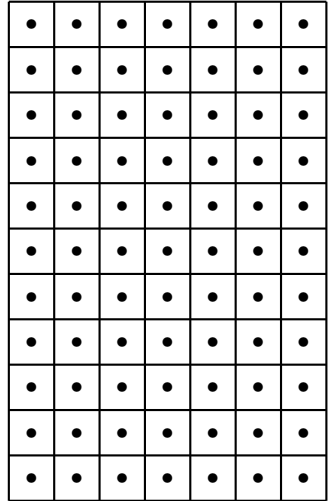
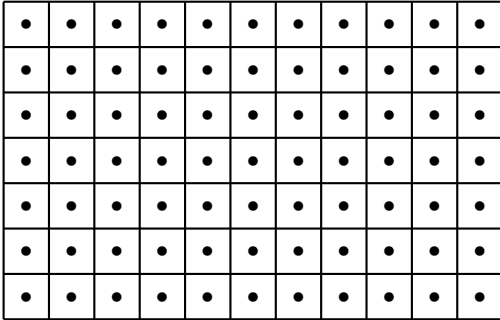
A Geometry of Points and Polygons

Oman Mathematics Committee seminar

David Pierce

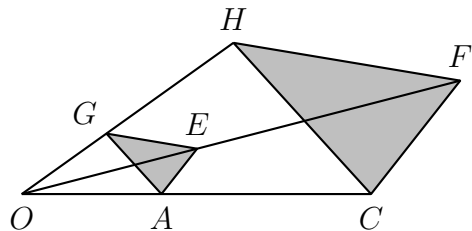
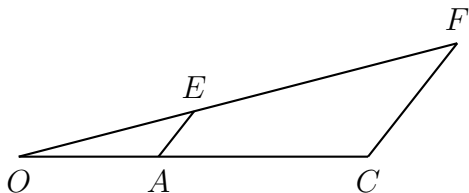
Mimar Sinan Güzel Sanatlar Üniversitesi
polytropy.com/mathematics/oman-2026/

April 22, 2026



Thales's Theorem (Euclid's VI.2)

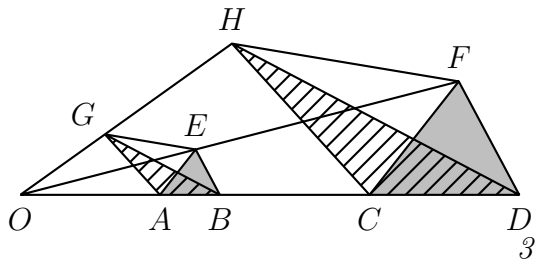
$$AE \parallel CF \iff OA : AC :: OE : EF \quad (1)$$



Desargues's Theorem

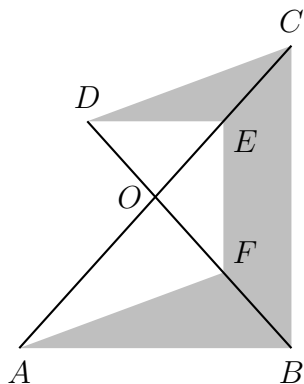
$$GA \parallel HC \ \& \ AE \parallel CF \implies GE \parallel HF. \quad (2)$$

$$ABE \sim CDF \ \& \ GE \parallel HF \implies ABG \sim CDH. \quad (3)$$

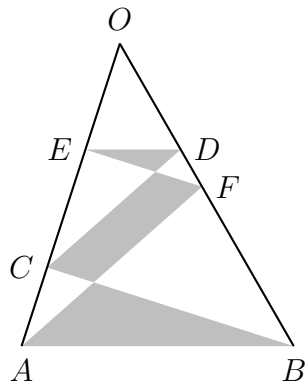


Hexagon Theorem

Pappus of Alexandria,
Collection ($\Sigma\Upsilon\Nu\Lambda\Gamma\Omega\Gamma\eta$), Book VII (~ 300 C.E.)



$$\begin{aligned} AB &\parallel DE \\ &\& \\ BC &\parallel EF \\ \implies \\ CD &\parallel FA \end{aligned}$$



Hessenberg,

“Beweis des Desarguesschen Satzes aus dem Pascalschen” (1905)

Geometry justifies algebra for Descartes in *La Géométrie* (1637).

He has been inspired by Pappus.

In model-theoretic terms, Descartes **interprets** a field K in a Euclidean plane. K could be \mathbb{R} , but need not be.

The plane can then be interpreted in K as K^2 .

Descartes's work was completed in the 19th and 20th centuries, in different ways, by Hilbert, Tarski, and Artin.

We shall show yet another way, using

- first-order logic (as Tarski does);
- **affine** geometry (as Artin does)—no circles;
- **ratios** (not lengths as Hilbert does) to constitute the field;
- **areas** in proofs, which then are **visual**.

Model theory

We shall axiomatize a theory of two *sorts*:

- an abelian group \mathcal{A} of **areas**;
- a set \mathcal{P} of **points**, with

$$(X, Y, Z) \mapsto XYZ : \mathcal{P}^3 \rightarrow \mathcal{A}, \quad (4)$$

so that A , B , and C are **collinear** if and only if

$$ABC = 0, \quad (5)$$

and then AB and CD are **parallel** if and only if

$$ABC \neq 0 \wedge CDA \neq 0 \wedge \forall Z (ABZ = 0 \Rightarrow CDZ \neq 0). \quad (6)$$

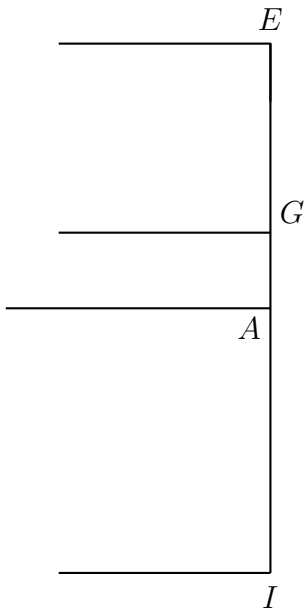
With an axiom

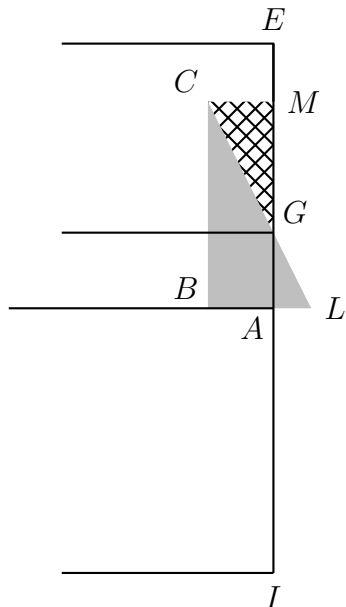
$$\forall t \exists X \exists Y \exists Z t = XYZ, \quad (7)$$

our theory will be bi-interpretable with the theory of fields of characteristic other than 2, nicely enough to be *companionable*.

A **five-line locus** problem of Descartes:

Given a vertical line
cut by four horizontal lines
at E , G , A , and I , where $EA = AI$,





A **five-line locus problem** of Descartes:

Given a vertical line
 cut by four horizontal lines
 at E , G , A , and I , where $EA = AI$,
 find C for which

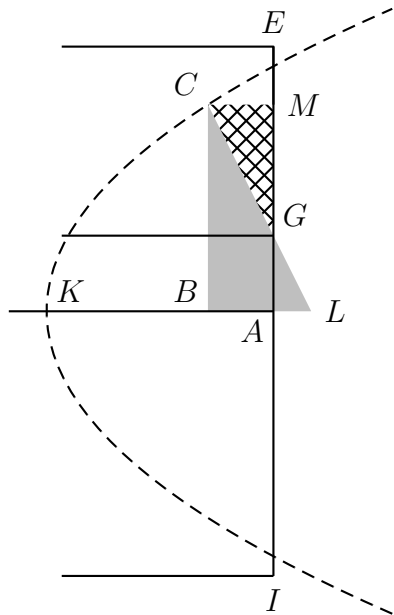
$$EM \cdot GM \cdot IM = AE \cdot MC \cdot AM. \quad (8)$$

Pappus would put it with ratios:

$$(EM : AE)(IM : AM) :: MC : GM. \quad (9)$$

By **similar triangles**,

$$LB : AM :: MC : GM. \quad (10)$$



Again, we *want* and *have*

$$(EM : AE)(IM : AM) :: MC : GM, \quad (11)$$

$$LB : AM :: MC : GM. \quad (12)$$

Thus we just need

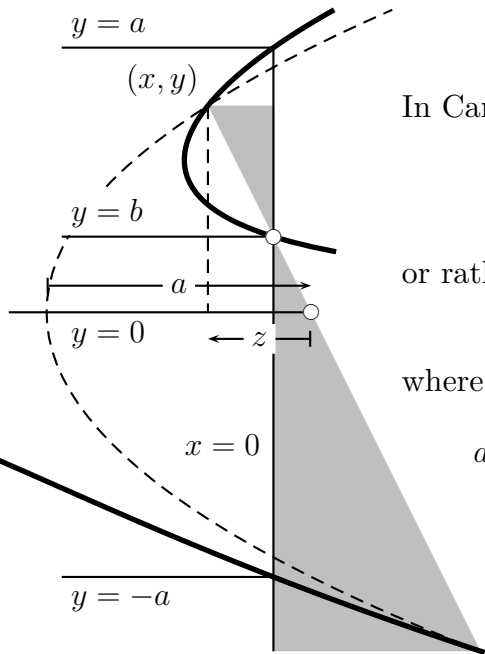
$$LB : IM :: EM : AE, \quad (13)$$

$$AE \cdot LB = IM \cdot EM = AM^2 - AE^2, \quad (14)$$

$$AE \cdot KB = AM^2, \quad (15)$$

where

$$KL = AE. \quad (16)$$



In Cartesian notation, we want

$$\frac{y-a}{a} \cdot \frac{y+a}{y} = \frac{x}{y-b}, \quad (17)$$

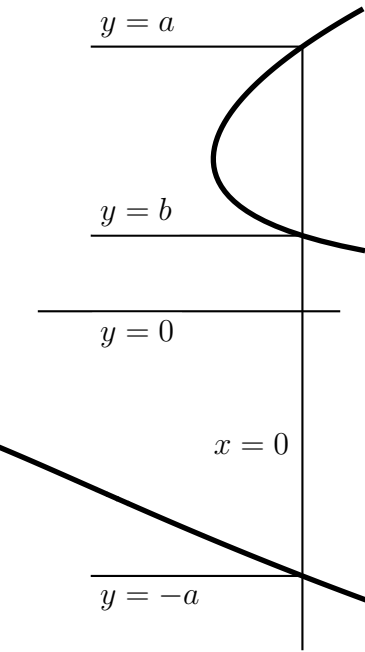
or rather

$$\frac{z}{y} = \frac{x}{y-b}, \quad (18)$$

where

$$az = (y-a)(y+a) = y^2 - a^2, \quad (19)$$

$$a(z+a) = y^2. \quad (20)$$



We have found the locus of (x, y) for which

$$y^3 - by^2 - a^2y + a^2b = axy. \quad (21)$$

Descartes: The **product** of any number of **lengths** as in (21) can be taken, once a **unit** is chosen.

Pappus: The product of any number of **ratios** can already be taken.

Wikipedia: “a ratio shows how many times one number contains another.” No! We have to show first that ratios can *serve* as numbers. For this, we go back to Euclid (~ 300 B.C.E.).

Euclid's *Elements*

Book I: *Equal* magnitudes in a plane: lengths, angles, areas.

Proposition 1. To construct an equilateral triangle.

Prop. 2 & 3. To cut from a ray a segment equal to another.

Prop. 4. “Side Angle Side” . . .

Prop. 47 & 48. Pythagorean Theorem and converse.

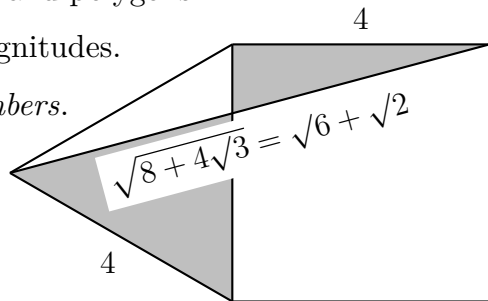
Books II–IV: Rectangles, circles, and polygons.

Books V & VI: *Proportional* magnitudes.

Books VII–IX: Proportional *numbers*.

Book X: Incommensurable areas.

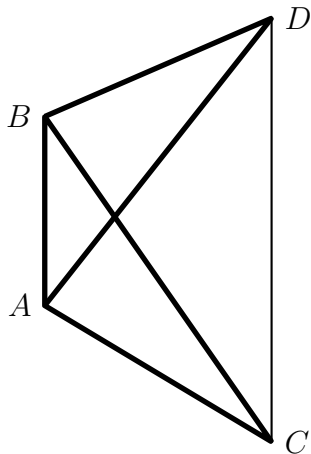
Books XI–XIII: Solid geometry.



Elements I, Propositions 37 & 39

Triangles sharing a base are equal
if and only if they have the same height.

Suppose $AB \parallel CD$.

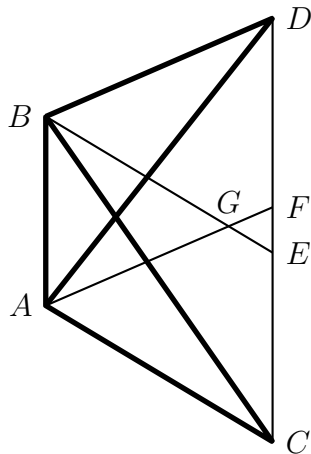


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1. Let $BE \parallel AC$ and $AF \parallel BD$.

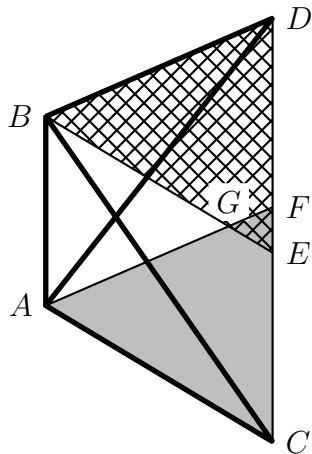


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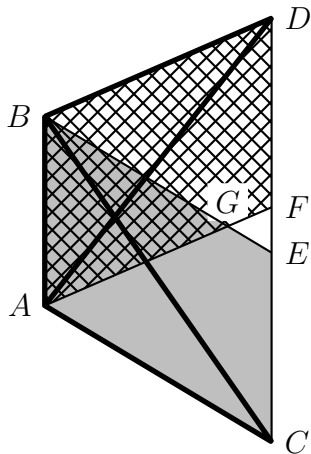


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Triangles sharing a base are equal
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1. Let $BE \parallel AC$ and $AF \parallel BD$.
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3. $AE = AFC - GFE + ABG$
 $= BDE - GFE + ABG = BF$.

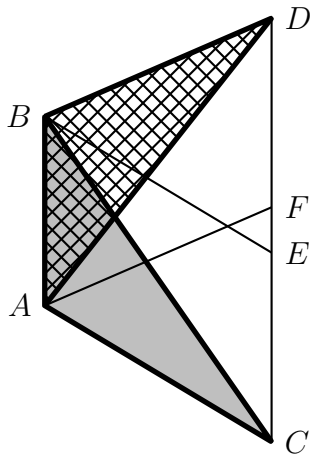


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2. $AFC = BDE$.
3. $AE = AFC - GFE + ABG$
 $= BDE - GFE + ABG = BF$.
4. $AE = 2ABC$ and $BF = 2ABD$.

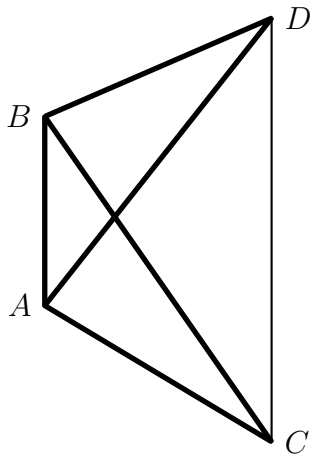


Elements I, Propositions 37 & 39

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2. $AFC = BDE$.
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 $= BDE - GFE + ABG = BF$.
4. $AE = 2ABC$ and $BF = 2ABD$.
5. $ABC = ABD$.

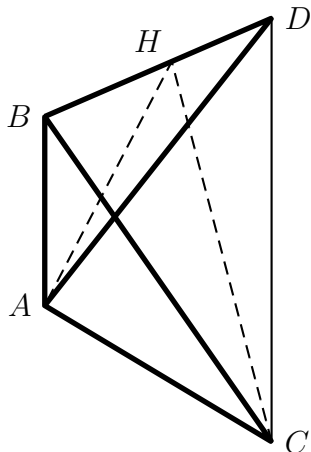


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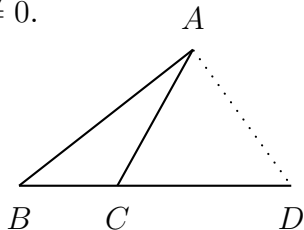
1. Let $BE \parallel AC$ and $AF \parallel BD$.
2. $AFC = BDE$.
3. $AE = AFC - GFE + ABG$
 $= BDE - GFE + ABG = BF$.
4. $AE = 2ABC$ and $BF = 2ABD$.
5. $ABC = ABD$.
6. $ABD > ABH$.



We have used only the following **affine** axioms, which give the theory already mentioned.

1. To a given line, through a given point, exactly one parallel passes.
2. Areas of triangles persist under parallel translation.
3. The areas of polygons compose an abelian group.
4. Diagonals bisect parallelograms.
5. The group of areas is torsion-free or has odd prime exponent.
6. $ABC \neq 0 \wedge BCD = 0 \wedge C \neq D \Rightarrow ACD \neq 0$.

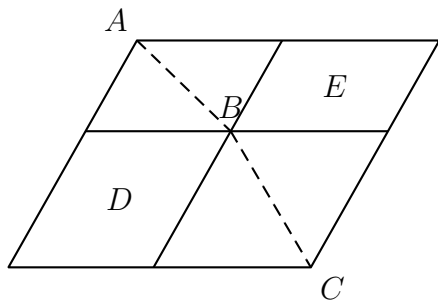
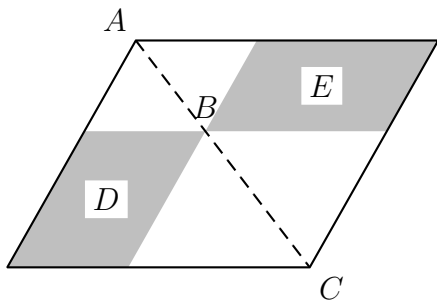
Next is another consequence.



Elements 1, Proposition 43

In a parallelogram,
the complements of parallelograms about a diagonal are equal.

The converse is also true.

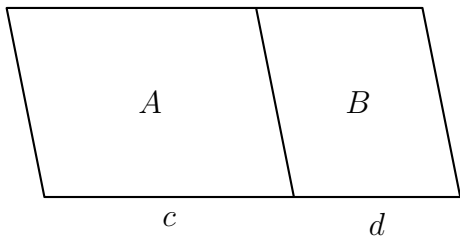


$$A \in BC \iff D = E. \quad (22)$$

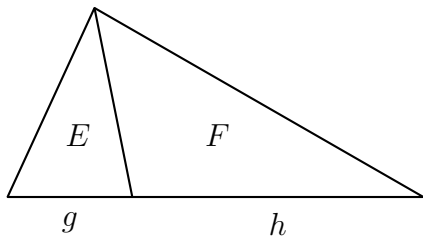
Elements VI, Proposition 1

By the definition of *proportion* or **sameness of ratio** from Book V:

A parallelogram or triangle of fixed height varies as its base.

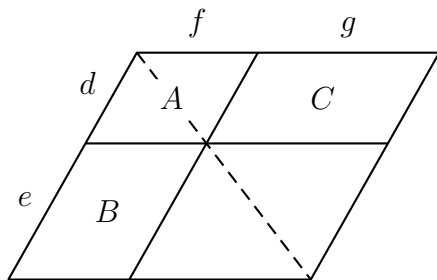


$$A : B :: c : d,$$



$$E : F :: g : h. \quad (23)$$

What is a ratio?

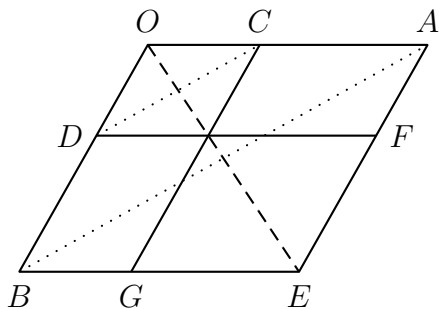


$$A : B :: d : e, \quad A : C :: f : g, \quad B = C. \quad (24)$$

It would seem then

$$d : e :: f : g. \quad (25)$$

What does this even mean?



The following will be equivalent.

$$DG = CF, \quad (26)$$

$$DC \parallel BA, \quad (27)$$

$$OC \cdot DB = OD \cdot CA. \quad (28)$$

The last is not affine.

Each will be equivalent to

$$OD : DB :: OC : CA, \quad (29)$$

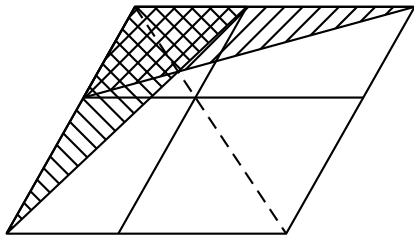
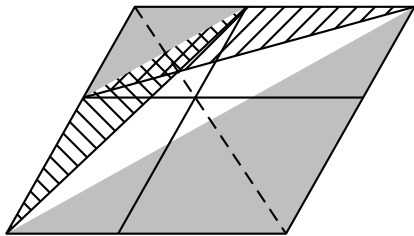
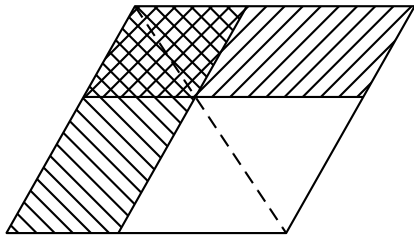
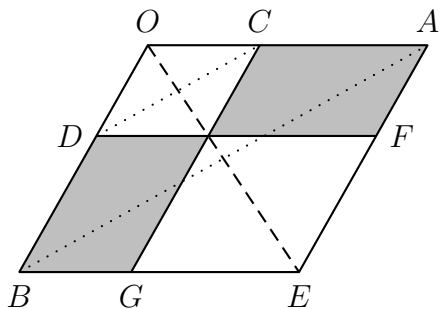
$$OD : OB :: OC : OA. \quad (30)$$

It cannot be so by definition—not without more work.

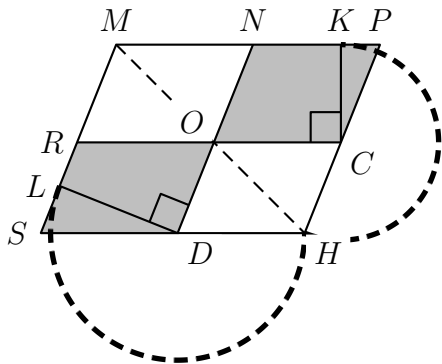
We saw (26) \iff (29), assuming *Elements* VI.1.

(27) \iff (29) is **Thales's Theorem**, *Elements* VI.2.

$$DG = CF \iff DC \parallel BA. \quad (31)$$



$$DG = CF \iff OC \cdot DB = OD \cdot CA. \quad (32)$$



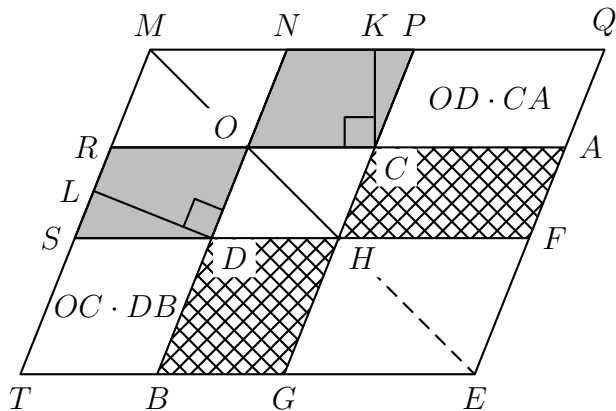
First,

$$M \in OH, \quad (33)$$

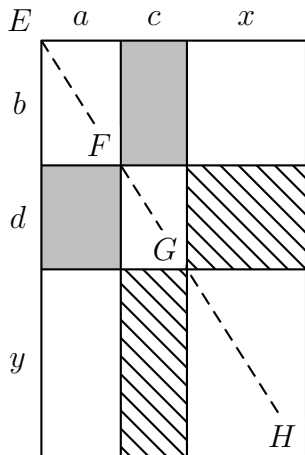
because

$$RD = OC \cdot OD = NC. \quad (34)$$

$$DG = CF \iff OC \cdot DB = OD \cdot CA. \quad (35)$$



Now we do the “more work” mentioned, of showing that one of the three equivalent conditions defines a *transitive* relation.



If

$$a \cdot d = b \cdot c, \quad c \cdot y = d \cdot x, \quad (36)$$

then

$$E \in FG, \quad F \in GH, \quad (37)$$

so

$$a \cdot y = b \cdot x. \quad (38)$$

Thus, defining

$$(a : b) = \{(x, y) : a \cdot y = b \cdot x\}, \quad (39)$$

we conclude

$$a : b :: c : d \iff a \cdot d = b \cdot c. \quad (40)$$

We compare with the “**Eudoxan**” definition in *Elements* v.

With numbers, we can understand

$$\ell : m > k : n \iff n\ell > km. \quad (41)$$

With magnitudes A and B then, we require

$$A : B > k : n \iff nA > kB. \quad (42)$$

Now Euclid's definition is effectively

$$A : B = \{(k : n) : nA > kB\}, \quad (43)$$

a **cut** of the “rationals.” Dedekind (1831–1916) defined cuts without geometry.

Unlike ours, the Eudoxan definition needs the “**Archimedean**” **postulate**: some multiple of the less magnitude exceeds the greater.

Whichever definition we use, we can **compose** ratios by

$$(a : b)(b : c) :: a : c. \quad (44)$$

Composition is **commutative**, because if

$$EF \parallel CB, \quad CD \parallel AF, \quad (45)$$

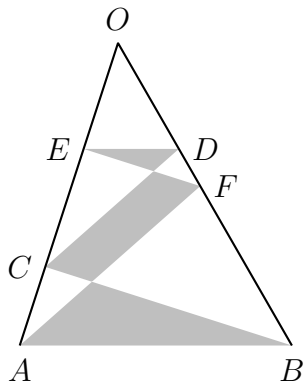
then

$$OE : OC :: OF : OB, \quad (46)$$

$$(OD : OF)(OE : OC) :: OD : OB, \quad (47)$$

and likewise

$$(OE : OC)(OD : OF) :: OE : OA. \quad (48)$$



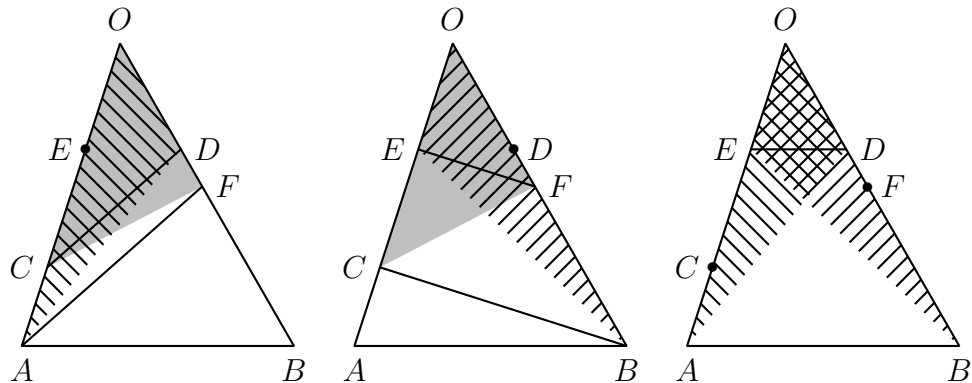
The two products are the same, because

$$AB \parallel ED, \quad (49)$$

by **Pappus's Hexagon Theorem**, applied to $ABCDEF$.

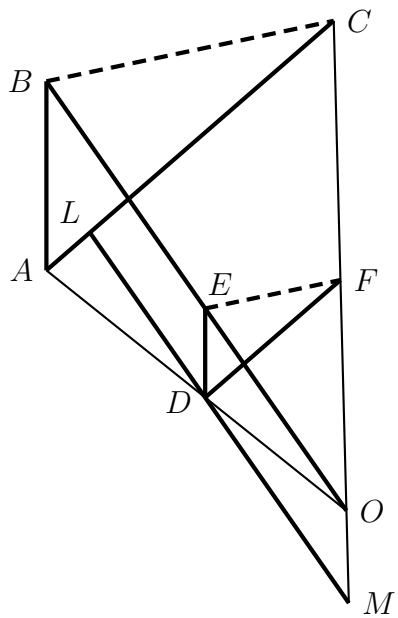
Pappus's Hexagon Theorem

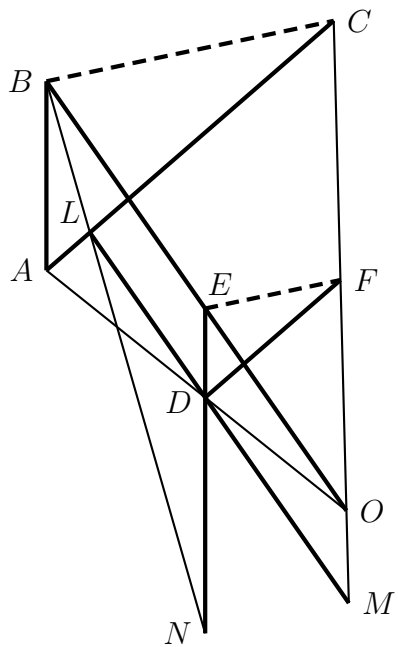
$$CD \parallel AF \text{ \& \ } EF \parallel CB \implies AB \parallel ED. \quad (50)$$

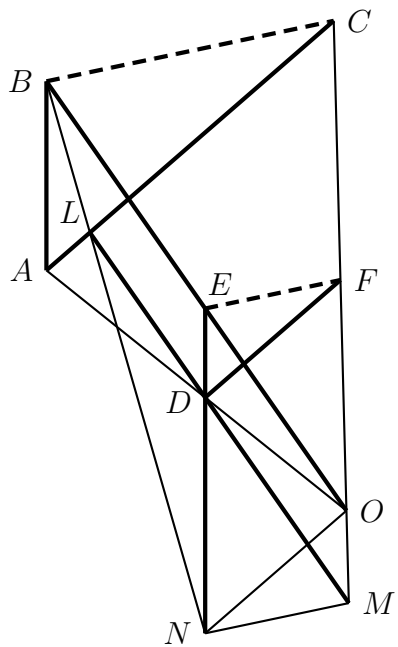


Ratios now compose a (commutative) field, acting on the plane. We needed only *Elements* I for this.

We did use circles, but are going to be able to do without them.

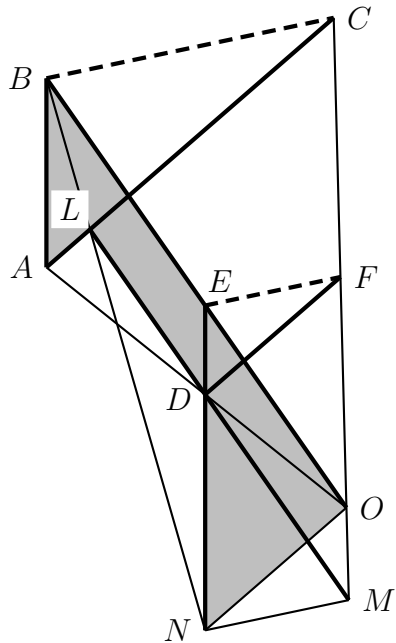


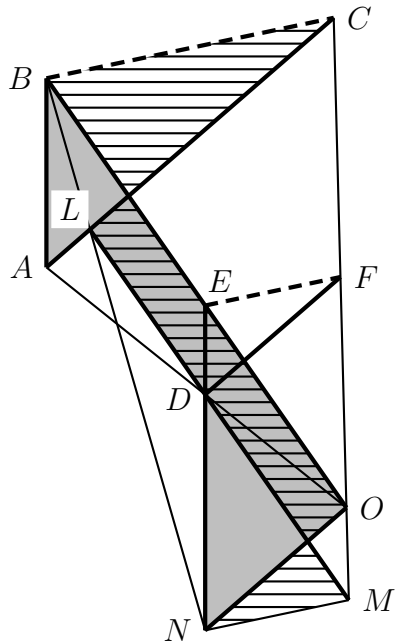




From *ONDLAB*,

$$ON \parallel AL \parallel DF.$$



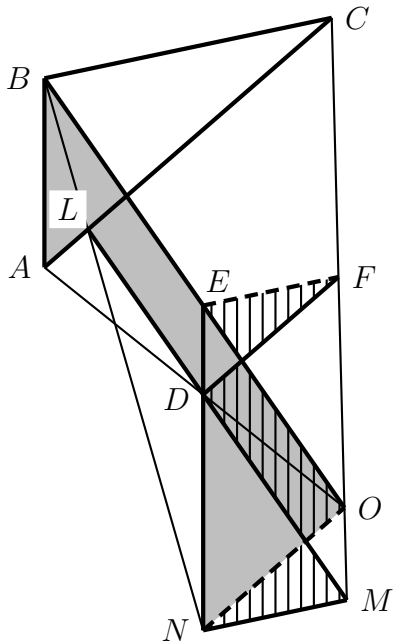


From $ONDLAB$,

$$ON \parallel AL \parallel DF.$$

From $ONMLCB$,

$$BC \parallel MN.$$



From $ONDLAB$,

$$ON \parallel AL \parallel DF. \quad (53)$$

From $ONMLCB$,

$$BC \parallel MN. \quad (54)$$

From $ONMDFE$,

$$EF \parallel MN \parallel BC. \quad (55)$$

Now, in place of our earlier definition of proportion, we can just use Thales, because Desargues then ensures transitivity of sameness of ratio.

Model theory

We can now use collinearity alone to axiomatize a theory of affine planes in which a field is interpretable.

A problem is that the union of an increasing chain of models may not itself be a model.

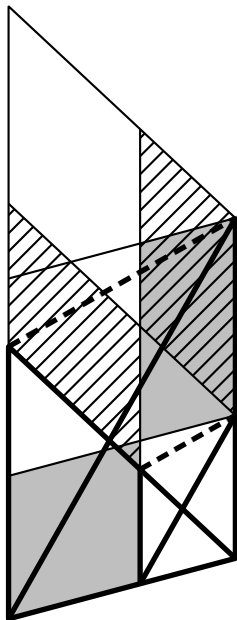
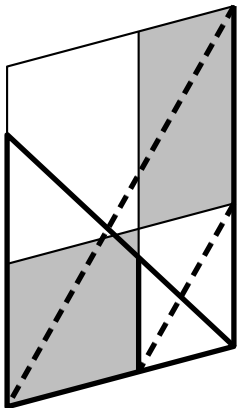
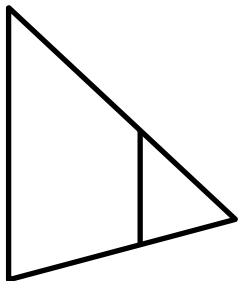
We can solve that problem by introducing a symbol for parallelism.

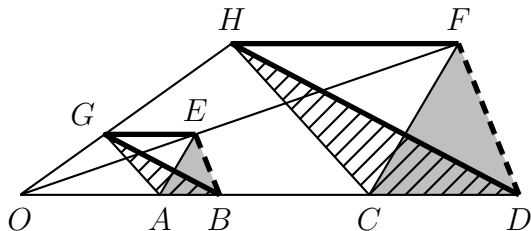
The theory is still “aesthetically” deficient for requiring Pappus as an axiom.

Using a sort of areas lets us use axioms closer to what Euclid and Pappus actually assume.

Meanwhile, there is an alternative proof of Desargues from Pappus.

First we can prove a special case.

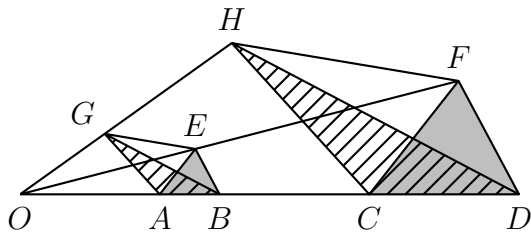


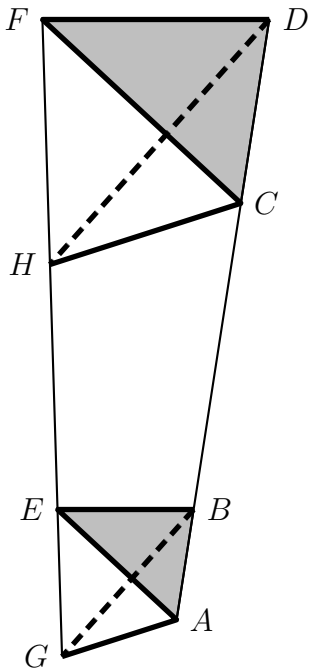


We want to get rid of the condition $OD \parallel GE$ here:

$$ABE \sim CDF \text{ \& } GE \parallel HF$$

$$\text{\& } OD \parallel GE \implies AGE \sim CHD. \quad (56)$$

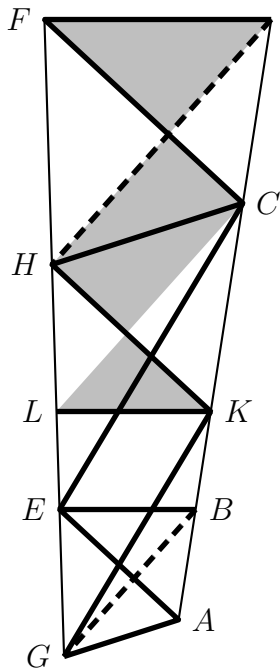




It is enough to show

$$\begin{aligned}
 ABE \sim CDF \quad & \& \quad AG \parallel CH \\
 & \implies ABG \sim CDH. \quad (57)
 \end{aligned}$$

Again, three applications of Pappus will do it.



Let

$$HK \parallel EA. \quad (60)$$

Then

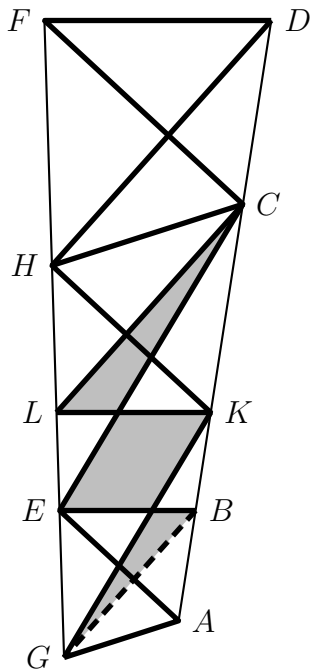
$$EC \parallel GK. \quad (61)$$

Let

$$LK \parallel FD. \quad (62)$$

Then

$$HD \parallel LC. \quad (63)$$



Let

$$HK \parallel EA. \quad (64)$$

Then

$$EC \parallel GK. \quad (65)$$

Let

$$LK \parallel FD. \quad (66)$$

Then

$$HD \parallel LC, \quad (67)$$

and also

$$LK \parallel EB, \quad (68)$$

and therefore

$$GB \parallel LC \parallel HD. \quad (69)$$

That's all; cheers!

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