On Hilbert's Third Problem

Whether there is a proof by cutting and pasting of the consequence of Euclid's *Elements*, Book XII, Proposition 5, whereby pyramids of the same height on equal bases are equal – there isn't

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Preface

This document comes out of working through Euclid's *Elements,* book by book, proposition by proposition, with a few devoted souls, since the winter of 2023. The original aim of the seminar was to get some sense of the mathematics that the guardians of the city would have to learn, by the account of Socrates in Plato's *Republic.*

The work of Euclid is still being done today: it is being not only kept alive, but also grown, furthered, augmented. Somehow, mathematics continues being the same science that it always was. I try to demonstrate this by basing my exposition of some modern mathematics on Euclid, as much as I can.

I cannot *always* do that. I do sometimes resort to Cartesian, algebraic notation. I aim to keep this to a minimum. The practice would seem to make my treatment of Hilbert's Third Problem unusual.

It was by composing the notes here that I worked out my own understanding of a solution to the Problem in question. I am thus confirming what must have been observed by many people, including now Rose

Horowitch in *The Atlantic* [12]:

> Writing is the way people figure out what they think, and how to convey those thoughts to someone who doesn't already share them. Cal Newport, a computer-science professor at Georgetown University, argues that the process of writing forces people to think in an orderly, linear fashion. It exposes flabby thoughts and shoddy reasoning. And the time and focus it takes to form thoughts into words, sentences, and paragraphs allow the author to make new connections and discover new insights.
>
> This feels true to me . . . The writing process is how I refine and formalize inchoate ideas and gain new understanding. By evaluating my arguments and discarding those that aren't convincing, I find the ones that are. Writing is hard because the writer is learning. If AI eliminates the challenge, it also eliminates the learning.

I did not know at the start that my notes would end up being so long. As I recall, I first constructed some of the diagrams, particularly in Figures 4, 5, 19, and 21.

In the last section, I end up doing theoretical linear algebra, without quite saying so. I try to avoid talking about "rational numbers," since Euclid does not really have that concept, unless we count a ratio of *his* numbers as one of *our* "rational numbers."

In the List of Tables, created automatically by the LaTeX program, I have not figured out why they are numbered 2 and 3, rather than 1 and 2.

Strictly speaking, I use `lualatex` for typesetting, so that Greek text can appear as such (in Unicode) in the underlying `tex` file. (Greek letters in mathematics are still control sequences in the `tex` file: α is `\alpha`, and so on.)

As for mathematics' being the same science it always was, I don't think the same can be said of physics or biology, not to mention chemistry. In an *empirical* science, as far as I can tell, new discoveries change the science, sometimes fundamentally. Once gravity has been identified as a force explaining the motions of both rain and the planets; once blood pumped *out* of a human heart is understood to return to it; once chemical reactions can be interpreted by means of a periodic table of elements: when these things have been done, there does not seem to be much point in pretending that they haven't, at least if one is training future scientists.

If I can be proved wrong on that point, I shall be glad, since I certainly enjoyed learning those discoveries as if they were new. Could there be a university *science* department where students read Aristotle, or Newton, or Lavoisier? Students in the department of mathematics where I work do read Euclid, optionally in the translation by a mathematician at a different university [20].

Mathematics seems peculiar in having students start out from more or less the same place that they always have: zero. In the journey thus begun, perhaps some shortcuts have been learned in the last twenty-three centuries. I am still trying to figure out how useful they really are.

Contents

1 **Introduction** 9

2 **Euclidean propositions** 24

3 **Hilbert's Third Problem** 39

4 **Reflection** 46

5 **Hill tetrahedra** 60

6 **Equilateral pyramids** 73

7 **Trigonometry** 79

8 **Wedges** 94

9 **Rational independence** 106

References 110

List of Figures

1 $A = \frac{1}{2}bh$ as a sum 12
2 $A = \frac{1}{2}bh$ as a difference 13
3 Parabolic segment and inscribed triangle 15
4 Triangular prism, cut from cube and into three pyramids 22
5 Three tetrahedra making up a triangular prism 23
6 *Elements* I.35 25
7 *Elements* I.36 28
8 *Elements* XI.31, non-perpendicular case 31
9 *Elements* XI.31, perpendicular case 32
10 Alternative proof for XI.31 33
11 *Elements* I.37 35
12 Equidecomposability of reflections without reflection . . 47
13 Another reflectionless reflection 49
14 Equicomplementability without reflection 50
15 Two triangles 55
16 Tetrahedron flattened 56

17 Tetrahedron unfolded 58
18 Tetrahedron cut open 61
19 Cube: plain and bisected three ways 63
20 Cube bisected three ways, diagonal view 64
21 Rotational symmetry 65
22 Rotational symmetry in color, steps 1 and 2 66
23 Rotational symmetry in color, steps 3 and 4 67
24 Tetrahedron re-arranged, steps 1 and 2 69
25 Tetrahedron re-arranged, steps 3 and 4 71
26 Tetrahedron re-arranged, alternative view 72
27 Cube with four tetrahedra cut off, one remaining 74
28 Same cube, rotated 75
29 Equal incongruent tetrahedra 76
30 Trigonometry 78
31 Ptolemy's Theorem 81
32 Proof of Ptolemy's Theorem 83
33 Angle addition 85

34 Angle subtraction 86
35 Dehn's "cylindrical plate" 96

List of Tables

2 Multiple angle formulas 88
3 Coefficients in multiple angle formulas 89

1 Introduction

When a proposition in Euclid's *Elements* concerns parallelograms, there is often an analogue concerning parallelepipeds. When a proposition about triangles is *derived* from a proposition about parallelograms, there may be an analogue about pyramids. In such a case, the *demonstrations* may not be analogous.

There is a reason for that, and after two thousand years, we can show it. The way calculus is difficult, some of Euclid's propositions are difficult: not because Euclid failed to find better proofs, but because there are none. We can prove *this.* That is what these notes are about.

In school today, we continue to learn both the mathematics of Euclid and the physics of Newton. However, it seems the Einstein-First Project in Australia would like to change the latter fact, saying on their webpage [4],

> The detection of gravitational waves in 2015 proved that Albert Einstein was correct, once again. The physics of our universe is Einsteinian – not Newtonian. Currently, school curricula [*sic*] *does not* include Einsteinian concepts and still contains restricted Newtonian physics. The Einstein-First Project aims to teach students the fundamental truths of modern physics and also improve student attitudes towards science.

Our universe is not Euclidean either. It is still "locally" Euclidean, as in another sense it is "locally" Newtonian. In any case, Euclidean geometry remains as *mathematically* correct as it always was. It is also a site for new discovery.

People today may think they can give a better account than Euclid did of Euclidean geometry. Perhaps they can, depending on who is listening. Still, unlike Kepler's laws of planetary motion, at least if you want to send a ship to Pluto, the propositions of Euclid are not in need of any refinement. Even the proofs are no harder than need be. This is what we are going to show, for one notorious case.

In school today, one learns that

- the area of a triangle is a half of base times height;
- the volume of a pyramid is a third of base times height.

The factor of one-half for the triangle can be established as in Figure 1 or 2, where the shaded triangle is the sum or difference of halves of two rectangles, and the sum or difference of the rectangles themselves is a rectangle whose area is bh.

As for the factor of one-third for the pyramid, it may be made *plausible* as in Figures 4 and 5, on pages 22 and the following, where a cube is divided into two congruent triangular prisms, and then one of the prisms is divided into three pyramids, each of which has, with

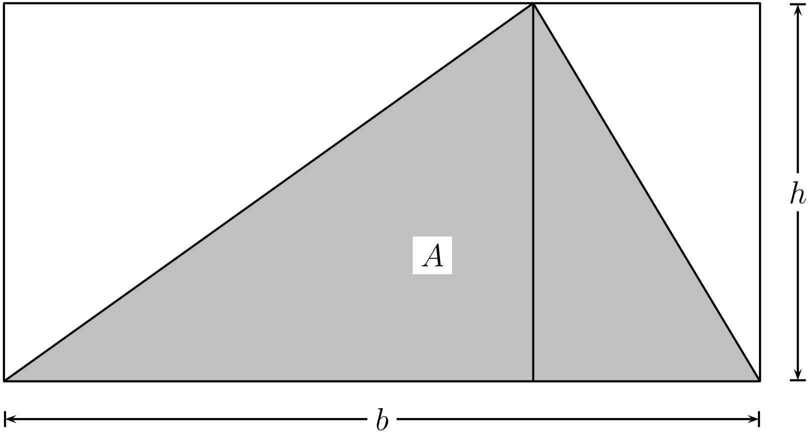

Figure 1: $A = \frac{1}{2}bh$ as a sum

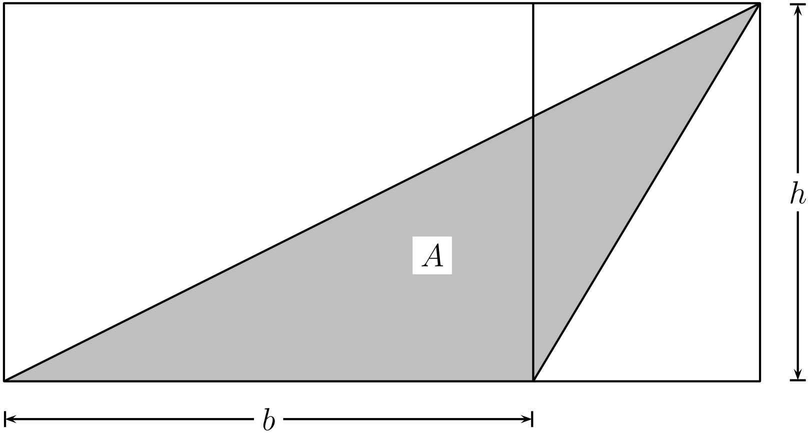

Figure 2: $A = \frac{1}{2}bh$ as a difference

respect to each of the other two,

- a base congruent with a base, and
- the same height (with respect to that base).

If that means the pyramids are all equal to one another, then each one has a third of the volume of the prism.

However, even though no curves are involved, Euclid's *proof* that the pyramids are all equal is at the level of Archimedes's proof that a segment of a parabola cut off by a straight line, as in Figure 3, exceeds by a third the triangle that is inscribed so that its base is parallel to the tangent to the parabola at the apex of the triangle. The proofs are by what has been called the method of exhaustion since the seventeenth century [15, p. 83].

We are not going to give those proofs. We are going to investigate why Euclid's, at least, has to be difficult.

Euclid does not actually write out formulas such as

$$A = \frac{1}{2}bh, \qquad\qquad V = \frac{1}{3}Bh,$$

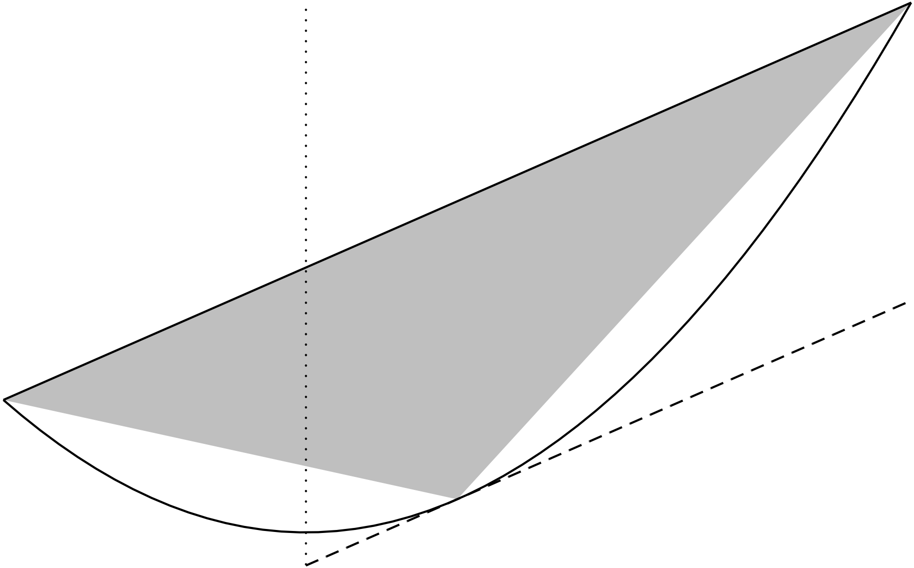

Figure 3: Parabolic segment and inscribed triangle

but he establishes the equivalent. First he has to show that such formulas are *possible.* For example, **Proposition 37 of Book I** is that triangles with the same height on the same base are equal. It may be hard to see why this *needs* a proof, if one has already been trained to apply the rule $A = \frac{1}{2}bh$. In the next section, starting on page 24, we are going to review the Euclidean proposition and more, including most of the other propositions mentioned – first time in bold – in *this* section.

Euclid does *not* explicitly prove the analogue of I.37 whereby pyramids with the same height on the same triangular base are equal. He does *use* the analogue. This analogue is a consequence of **Proposition 5 of Book XII,** that pyramids on triangular bases and with the same height are to one another as the bases – that is, the pyramids are in the same *ratio* as the bases, or they *vary* as the bases.

That proposition is the analogue of the part of **Proposition 1 of Book VI** whereby triangles of the same height are to one another as their bases.

The *proof* of XII.5 is not analogous to that of the analogous part of VI.1.

To prove that part, we can use I.37 to show that a multiple of a triangle on a certain base is equal to a triangle on the same multiple of that base.

Proposition I.37 is true, because

1) parallelograms on the same base and with the same height are equal, by **Proposition 35 of Book I,** and
2) a diameter or diagonal of a parallelogram divides it into two triangles that are equal, because they are congruent, by **Proposition 34 of Book I.**

We have an analogue of I.35 in **Propositions 29 and 30 of Book XI,** whereby parallelepipeds on the same base and with the same height are equal.

We are going to talk a lot about congruence. I understand it to be what Euclid is talking about in one of his common notions, here with Heiberg's Latin translation [5, pp. 10f.], then Heath's English [6, p. 224]:

Τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

Quae inter se congruunt, aequalia sunt.

Things which coincide with one another are equal to one another.

Where Heath has "coincide," I would say "are congruent," as I do elsewhere [17, p. 106]. Heath himself suggests on his next page,

> It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an *axiom of congruence.*

In the paper just alluded to, I say, "An isosceles triangle is symmetric because it is congruent to its mirror image" [17, p. 92]. More precisely, reflecting an isosceles triangle about the perpendicular drawn from the apex to the base has no effect on the area filled out by the triangle, and the effect of reflecting the triangle about any other axis can be achieved by translation or rotation.

I would say now that *every* triangle is *automatically* congruent to its reflection, by the criterion applied in the demonstration of **Proposition 4 of Book I.** Since a triangle has two sides equal to two sides of its mirror image, and the included angles are equal, the one triangle can be applied to the other so that they coincide, that is, are congruent. In short, reflecting a triangle *is* applying it to a mirror image.

Euclid's first application of I.4 is the very next proposition, that in an isosceles triangle, the angles at the base are equal to one another, whether they be the internal or the external angles. The equal sides of the triangle being AB and AC, if these are extended equally to D and E respectively, then the triangles ABE and ACD have two sides equal to two sides and the included angles equal, and the triangles are therefore congruent by I.4, even though they are reflections of one another.

Ultimately it does not matter whether a figure is congruent with its mirror image by *definition*; even if figures can be congruent only by translation and rotation, every polygon and every polyhedron is equal to its reflection, as we shall see in § 3.

Meanwhile, we said that I.37, which is that triangles with the same height on the same base are equal, was a consequence, both of I.35, which says the same for parallelograms, and of I.34, that a parallelogram is bisected by a diagonal; and I.35 has a parallelepipedal analogue. There is no such analogue for I.34, at least not fully. Coming *after* XII.5, that triangular pyramids vary as their bases, there is **Proposition 7 of Book XII,** that a prism on a triangular base can be divided into three *equal* pyramids on triangular bases. Two of those pyramids have the two triangular bases of the prism as their own bases, while having the same height as the prism. However, those two pyramids may not be *congruent.*

We shall show moreover that those two pyramids, which are equal by XII.5, may be indivisible into respectively congruent parts, even after first being augmented with respectively congruent solids.

We may speculate on how well Euclid understood this. It seems a *proof* was not generally known until the twentieth century.

Specifically, in Figure 4, a cube is bisected by the rectangle $ACDF$, and then the back half of the cube is divided, by the triangles ACH and ADH, into the three tetrahedra shown separately in Figure 5. If we accept the results already discussed, then the pyramid on rectangular base $ACDF$ with apex H is bisected by ADH, and thus we have an equation of tetrahedra,

$$ADFH = ACDH.$$

Likewise, the pyramid on square base $BCDH$ with apex A is bisected by ACH, and thus

$$ACDH = ABCH.$$

We now have three equal tetrahedra. We are going to show that they are not *equidecomposable* or even *equicomplementable.* We shall have defined those terms while reviewing Euclid in the next section. We pick up the thread in the nineteenth century in § 3, starting on page 39.

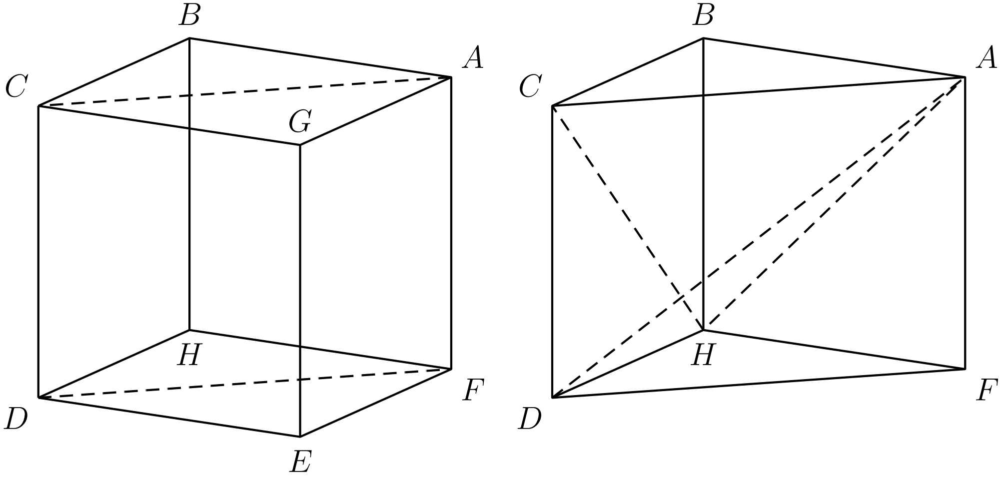

Figure 4: Triangular prism, cut from cube and into three pyramids

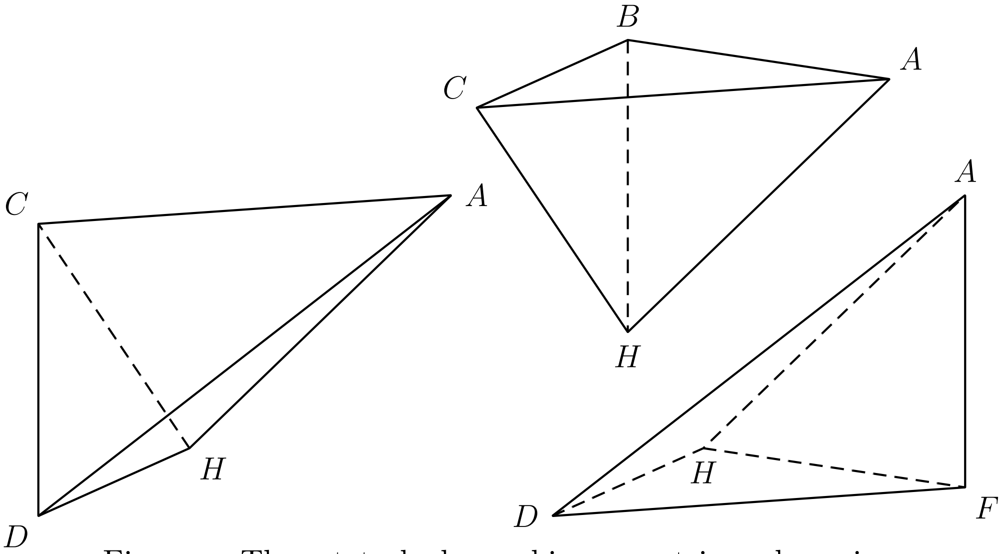

Figure 5: Three tetrahedra making up a triangular prism

2 Euclidean propositions

Book I, Proposition 35

Τὰ παραλληλόγραμμα	Parallelograms
τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα	on the same base
καὶ ἐν ταῖς αὐταῖς παραλλήλοις	and in the same parallels
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Proof. The parallelograms being $ABCD$ and $ABEF$ as in Figure 6, the trapezoid $ABED$ can be analyzed into parallelogram and triangle in two ways; in particular,

$$ABCD + CBE = DAF + ABEF.$$

The two triangles being congruent and therefore equal, the parallelograms must also be equal (albeit not congruent). □

For the foregoing proof, unlike Euclid's, the order of points on DE doesn't matter. We have shown that the two parallelograms are

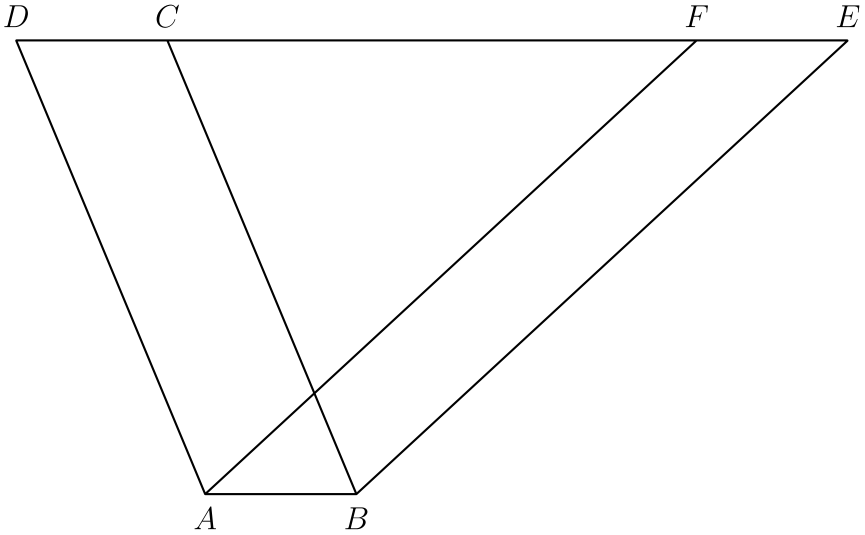

Figure 6: *Elements* I.35

equicomplementable: when congruent figures are added to them, the sums are figures that are congruent (as here) or at least *equidecomposable* in the sense defined below, after our proof of I.37.

In the same way, we can prove the same thing for parallelepipeds, but it needs two steps, one for each of the two dimensions of the base:

Book XI, Proposition 29

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα	When on the same base,
στερεὰ παραλληλεπίπεδα	parallelepipedal solids
καὶ ὑπὸ τὸ αὐτὸ ὕψος,	also under the same height
ὧν αἱ ἐφεστῶσαι	whose uprights
ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν,	end up on the same straight lines
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Book XI, Proposition 30

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα	When on the same base,
στερεὰ παραλληλεπίπεδα	parallelepipedal solids
καὶ ὑπὸ τὸ αὐτὸ ὕψος,	also under the same height
ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν	whose uprights do not
ἐπὶ τῶν αὐτῶν εὐθειῶν,	end up on the same straight lines
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Shifting from the *same* base to *equal* bases is pretty easy in two dimensions, more of a big deal in three:

Book I, Proposition 36

Τὰ παραλληλόγραμμα	Parallelograms
τὰ ἐπὶ ἴσων βάσεων ὄντα	on equal bases
καὶ ἐν ταῖς αὐταῖς παραλλήλοις	and in the same parallels
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Proof. The parallelograms being $ABCD$ and $EFGH$ as in Figure 7, Euclid's proof uses that the base AB is congruent, not only to its

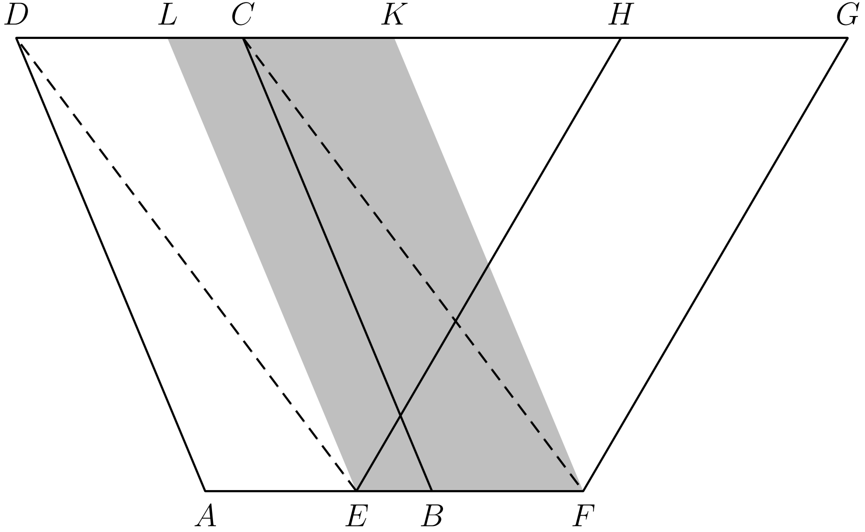

Figure 7: *Elements* I.36

opposite, DC, but to the base EF of the other parallelogram. Thus DC and EF are congruent (as well as parallel) and are therefore the opposite sides of a new parallelogram, which is equal to each of the first two by I.35.

Alternatively, we can construct $EFKL$ to be congruent to $ABCD$, and then it is equal to $EFGH$ by I.35. $\square$

Book XI, Proposition 31

Τὰ ἐπὶ ἴσων βάσεων ὄντα	When on equal bases,
στερεὰ παραλληλεπίπεδα	parallelepipedal solids
καὶ ὑπὸ τὸ αὐτὸ ὕψος	also under the same height
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Proof. Euclid's proof uses that magnitudes having the same ratio to the same are equal. We can instead obtain that the parallelepipeds on equal bases are *equicomplementable* by applying Propositions XI.29 and 30 along with I.43 (that the complements of parallelograms about the diagonal of a parallelogram are equal) and XI.28 (that a parallelepiped is halved by a plane containing diagonals of opposite faces).

Specifically, if (in Euclid's letters) the parallelepipeds AE and $\mathit{\Gamma} Z$ stand at the same height on the equal bases AB and $\mathit{\Gamma\Delta}$, but, as in Figure 8, the uprights are *not* (necessarily) at right angles to the bases, we can still assume that they *are* at right angles, by XI.29 and 30. This puts us in the situation of Figure 9, where we construct the copy $P\mathit{\Phi}$ of AE (which is also $\mathit{\Lambda} K$), and this copy is equal to $T\varsigma$ by XI.29. For an alternative to Euclid's proof now, in Figure 10, since $\mathit{\Omega} T = \mathit{\Gamma\Delta}$, the point P lies on $\alpha\aleph$, so the plane (or rectangle) $I\alpha\aleph\beth$ bisects the parallelepipeds OQ, $\mathit{\Delta\Psi}$, and $\mathit{\Gamma}\varsigma$.

The parallelepipeds are bisected, not only into *equal* prisms, but into *congruent* ones. Thus, we can add congruent prisms to $T\varsigma$ and $\mathit{\Gamma} Z$, and the results are congruent prisms. This gives us that $T\varsigma$ and $\mathit{\Gamma} Z$, and therefore AE and $\mathit{\Gamma} Z$, are equicomplementable. $\square$

When we come to triangles, analogies with solids break down, at least in the proofs.

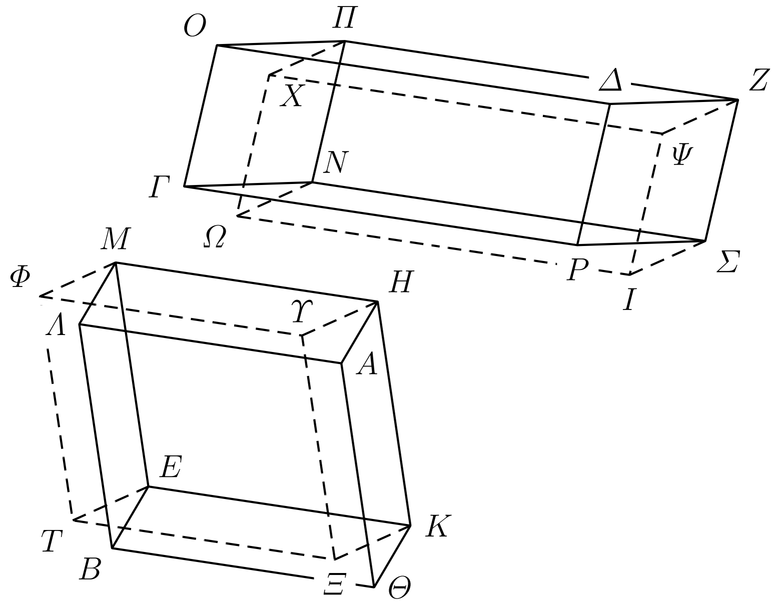

Figure 8: *Elements* XI.31, non-perpendicular case

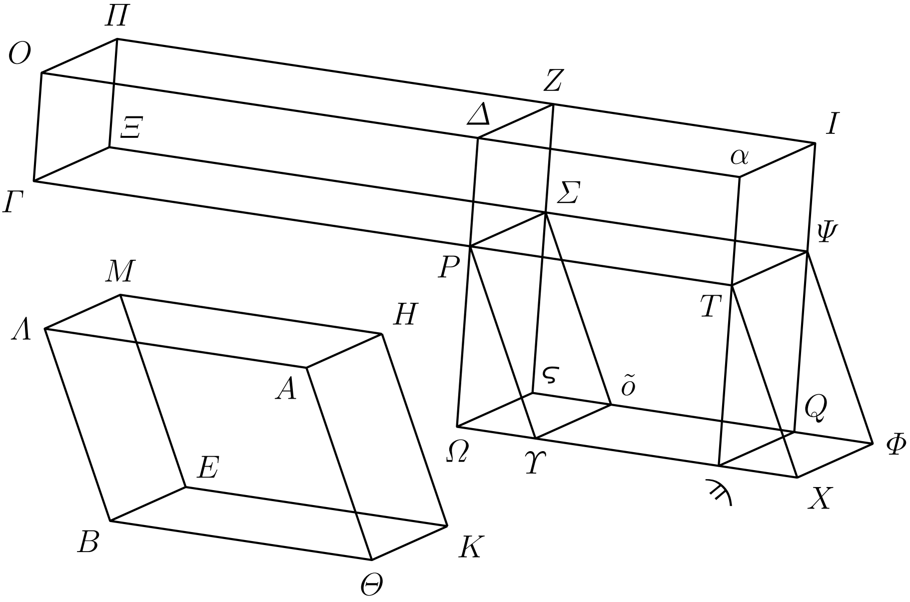

Figure 9: *Elements* XI.31, perpendicular case

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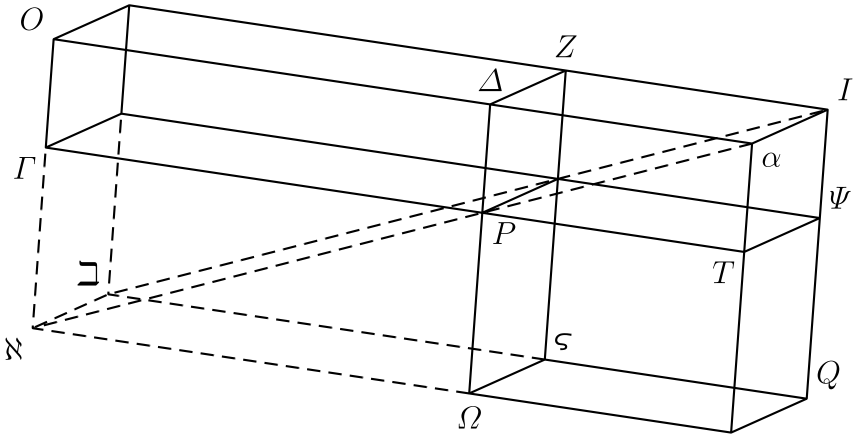

Figure 10: Alternative proof for XI.31

Book I, Proposition 37

Τὰ τρίγωνα	Triangles
τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα	on the same base
καὶ ἐν ταῖς αὐταῖς παραλλήλοις	and in the same parallels
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Proof. We can derive this from I.35, as Euclid does, using that the diagonal bisects the parallelogram, and halves of equals are equal. Alternatively, in Figure 11, the midpoint of AC being E; of BD, F; the parallelograms $ABKE$ and $ABFL$ being completed, they are equal by I.35. Moreover, since CEG and BKG are congruent,

$$ABC = ABGE + CEG = ABGE + BKG = ABKE.$$

Likewise, $ABFL = ABD$. □

We have just shown that ABC and $ABKE$ (and likewise $ABFL$ and ABD) are **equidecomposable:** divisible into respectively congruent (if not identical) figures.

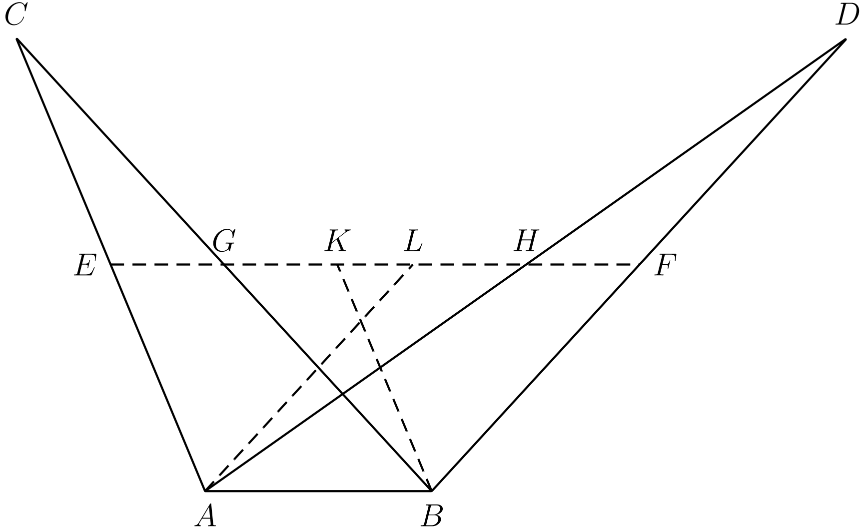

Figure 11: *Elements* I.37

Book I, Proposition 38

Τὰ τρίγωνα	Triangles
τὰ ἐπὶ ἴσων βάσεων ὄντα	on equal bases
καὶ ἐν ταῖς αὐταῖς παραλλήλοις	and in the same parallels
ἴσα ἀλλήλοις ἐστίν.	are equal to one another.

Proof. We can derive this from I.36, or else use I.37 after copying one triangle onto the base of the other. □

Pyramids on the same or equal triangular bases under the same height are also equal, but Euclid does not prove this separately; it is rather a consequence of the three-dimensional analogue of the part of following about triangles:

Book VI, Proposition 1

Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα,	Triangles and parallelograms
τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα	under the same height
πρὸς ἄλληλά ἐστιν	are to one another
ὡς αἱ βάσεις.	as the bases.

Proof. For parallelograms, this is easy, because we can understand a multiple of a parallelogram as a new parallelogram under the same height on the same multiple of the base. A multiple of a *triangle* is *equal,* by I.37, to a new triangle with the same apex and on the same multiple of the base. □

The parallelogrammatic part of the foregoing has an easy analogue in three dimensions.

Book XI, Proposition 32

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα	When under the same height,
στερεὰ παραλληλεπίπεδα	parallelepipedal solids
πρὸς ἄλληλά ἐστιν	are to one another
ὡς αἱ βάσεις.	as the bases.

The three-dimensional analogue of the triangular part of VI.1 may be the second non-elementary result in the *Elements,* not counting Book V. I mean, the demonstration involves the equivalent of what today is taking *limits* in calculus. The first such demonstration would be of Proposition 2 of Book XII, that circles are to one another as the squares on the diameters.

Book XII, Proposition 5

Αἵ ὕπὸ τὸ αὐτὸ ὕψος οὖσαι	When under the same height,
πυραμίδες	pyramids
καὶ τριγώνους ἔχουσαι βάσεις	having also triangular bases
πρὸς ἀλλήλας εἰσὶν	are to one another
ὥς αἵ βάσεις.	as the bases.

Again, we are not going to prove that, but show that something like it is needed for the following, whose proof we reviewed in the previous section.

Book XII, Proposition 7

Πᾶν πρίσμα	Every prism
τρίγωνον ἔχον βάσιν	that has a triangular base
διαιρεῖται	is divided
εἰς τρεῖς πυραμίδας	into three pyramids,
ἴσας ἀλλήλαις	equal to one another,
τριγώνους βάσεις ἐχούσας.	that have triangular bases.

3 Hilbert's Third Problem

Whether there could be an elementary proof of the foregoing XII.7 was an active question in the 19th century. At the end of that century, answering the question was one of the challenges that David Hilbert gave to the next century, in "Mathematical Problems: lecture delivered before the International Congress of Mathematicians at Paris in 1900" [9] (the footnotes, including the bracketed parts, are in the original):

> 3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes

In two letters to Gerling, Gauss[1] expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, *i.e.*, in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved.[2] Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent

[1]Werke, vol. 8, pp. 241 and 244.

[2]Cf., beside earlier literature, Hilbert, Grundlagen der Geometrie, Leipzig, 1899, ch. 4. [Translation by Townsend, Chicago, 1902.]

tetrahedra.[3]

The same passage is quoted by Boltianskii [2, pp. 47f.].

In the terms that we have introduced, which Hilbert defines for plane geometry in the section of *Grundlagen der Geometrie* called "Equidecomposability and Equicomplementability of Polygons" [10, § 18, pp. 60f.], Hilbert's Third Problem is to find pyramids on equal triangular bases and under the same height that are not equidecomposable or even equicomplementable. Dehn's solution is in "Ueber den Rauminhalt" [3].

As Hilbert observes in the *Grundlagen* [10, pp. 62f.], triangles on the same base in the same parallels can fail to be equidecomposable if (as he shows we may) we reject the "Archimedean" postulate that Euclid introduces implicitly in *Elements* V. By this postulate, the greater of

[3]Since this was written Herr Dehn has succeeded in proving this impossibility. See his note: "Ueber raumgleiche Polyeder," in *Nachrichten d. K. Gesellsch. d. Wiss. zu Göttingen*, 1900, and a paper soon to appear in the *Math. Annalen* [vol. 55, pp. 465-478].

two lengths, or areas, or volumes, is always exceeded by some multiple of the less. In Figure 6 (back on page 25), if no multiple of AB or of the height of ABC exceeds CE in length, then the triangles ABC and ABE are not equidecomposable. We proved that they were equicomplementable, without using the Archimedean postulate.

In *Dissections: Plane and Fancy* (1997), which is chock full of pictures of what we are calling equidecomposable figures, Frederickson writes about the correspondence that Hilbert mentions [7, pp. 230f.],

> In an exchange of letters between Carl Friedrich Gauss and Christian Ludwig Gerling in 1844, Gauss had lamented that the only known procedure for proving the equality of the volume of a polyhedron with the volume of its mirror image was a method of exhaustion. However, in his return letter, Gerling described a simple proof that established that an irregular tetrahedron has the same volume as its mirror image. Since a polyhedron can be split into tetrahedra, the result followed.

We argued in § 1 that for Euclid, a figure is automatically congruent with, and therefore equal to, its mirror image. It seems Gauss and his former student Gerling do not make this assumption. Gauss's letter is dated April 8, 1844 [8, pp. 241f.]:

> Ihre Bemerkungen über Symmetrie und Congruenz sind vollkommen treffend. Was noch zu desideriren wäre, ist der metaphysische Grund, warum es so ist (was bei Ihnen nur als eine wahrgenommene Thatsache auftritt) und damit auch die Erweiterung auf eine Geometrie von mehr als 3 Dimensionen, wofür wir menshliche Wesen keine Anshauung haben, die aber in abstracto betrachtet nicht widersprechend ist, und füglich höhern Wesen zukommen könnte. Um aber, aus dieser Höhe, wieder auf die Erde herunterzukommen, so ist es schade, dass die Gleichheit der Volumina körperlicher bloss symmetrischer, aber nicht congruenter Gebilde, sich nur durch die Exhaustionsmethode, und nicht eben so elementarisch demonstriren lässt, wie meines Wissens zuerst Sie bei der Area des sphärischen Dreiecks gezeigt haben. Ihr Beweis in etwas anderer Gestalt findet sich auch bei Lobatschewsky, ohne dass erhellt, ob er Ihren Aufsatz gekannt habe.

I don't read German, but the translation by Google may be good enough:

> Your remarks on symmetry and congruence are entirely accurate. What still needs to be explored is the metaphysical reason why this is so (which you present only as a perceived fact), and thus also the extension to a geometry of more than three dimensions, for which we human beings have no conception, but which, considered abstractly, is not contradictory and could justifiably apply to higher beings. However, to bring us back down to earth from this lofty perspective, it is unfortunate that the equality of the volumes of corporeal, merely symmetrical but not congruent, structures can only be demonstrated through the method of exhaustion, and not as elementary [*sic*] as you, to my knowledge, first showed with the area of the spherical triangle. Your proof, in a slightly different form, can also be found in LOBACHEVSKY, though it is unclear whether he was familiar with your essay . . .

I'm guessing that "elementary" ought to be "elementarily." Etymologically speaking, "symmetric" figures could be *commensurable,* not just in the sense of being measured *by* the same magnitude, but in the sense of *having* the same measure, that is, of being equal in the sense of Euclid. However, in his response on April 15, Gerling refers to mirror symmetry:

> Lehrsatz: Zwei beliebige symmetrische Polyeder sind von gleichem körperlichen Inhalt.
>
> Theorem: Any two symmetrical polyhedra have the same physical content.

We shall look at the proof in § 4. Meanwhile, Gauss writes again on April 17, concluding,

> Mein Bedauern muss ich nun, da jener Satz nicht mehr davon getroffen ist, auf die andern Sätze der Stereometrie beschränken, die annoch von der Exhaustionsmethode abhängig sind wie Euklid XII. 5. Vielleicht ist auch hier noch manches zu verbessern; in diesem Augenblick habe ich nicht Zeit, dem Gegenstande weiteres Nachdenken zu widmen.
>
> Since that proposition is no longer affected by it, I must now limit my regret to the other propositions of stereometry that still depend on the method of exhaustion, as in Euclid XII.5. Perhaps there is still much to improve here as well; at this moment I do not have time to devote further thought to the subject.

Dehn will show that Gauss's regret cannot be dispelled, as we shall see in § 8.

4 Reflection

Because it is interesting, and to gain experience, let us establish Gerling's result, though we do not strictly need it in order to solve Hilbert's Third Problem.

Gerling's theorem has a two-dimensional analogue. To show that a *triangle* is equidecomposable with its reflection, *without* reflection of the parts, we can cut the triangle along the perpendiculars drawn to the sides from the *incenter.* The incenter is the center of the inscribed circle, and Euclid finds it in *Elements* IV, Proposition 4. Once we make the cuts, we have three quadrilaterals, each symmetric about a diagonal. When we turn over each of the quadrilaterals about its line of symmetry, as in Figure 12, then turn over the whole original triangle, it is as if we just slid around the quadrilaterals.

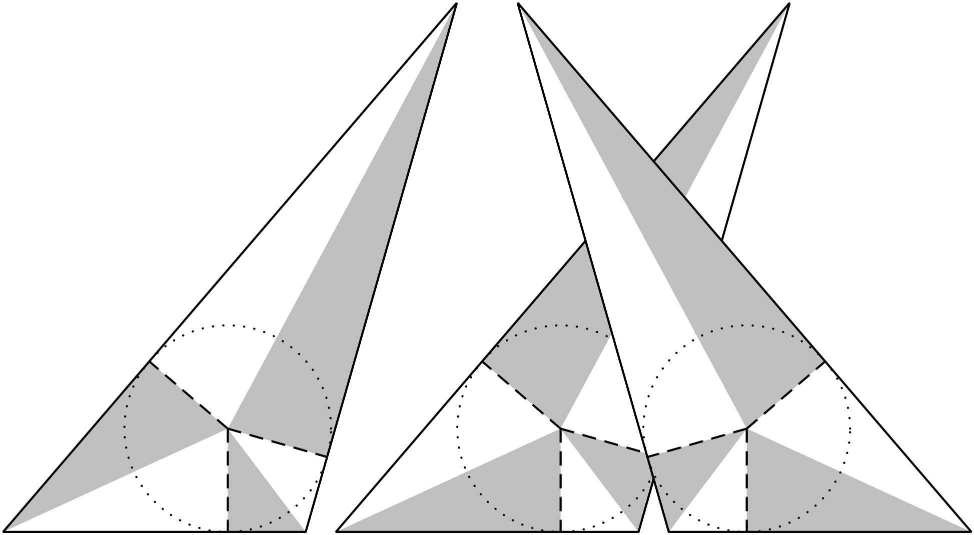

Figure 12: Equidecomposability of reflections without reflection

The incenter is the point, which must exist, where the bisectors of the three angles meet. I talk about this in [16], along with the circumcenter, which is found in VI.5 and is where the perpendicular bisectors of the the three sides of the triangle meet. By cutting from the circumcenter to the vertices, as in Figure 13, we can reflect the triangle without reflecting its parts – at least when none of the angles is obtuse. When there is an obtuse angle, as in Figure 14, then we are showing the *equicomplementability,* without reflection, of the triangle and its reflection.

Gerling carried the latter construction into three dimensions. Working now with a tetrahedron, if we take the circumcenter, K, of one of the faces, ABC, and we draw the perpendicular to ABC through K, then every point on that perpendicular is the center of a sphere containing each vertex of ABC. One of those spheres contains the fourth vertex, D, of the tetrahedron. The center, O, of *that* sphere is the circumcenter of the tetrahedron. As we can rearrange the isosceles triangles AKB, BKC, and CKA to get the mirror image of ABC, so

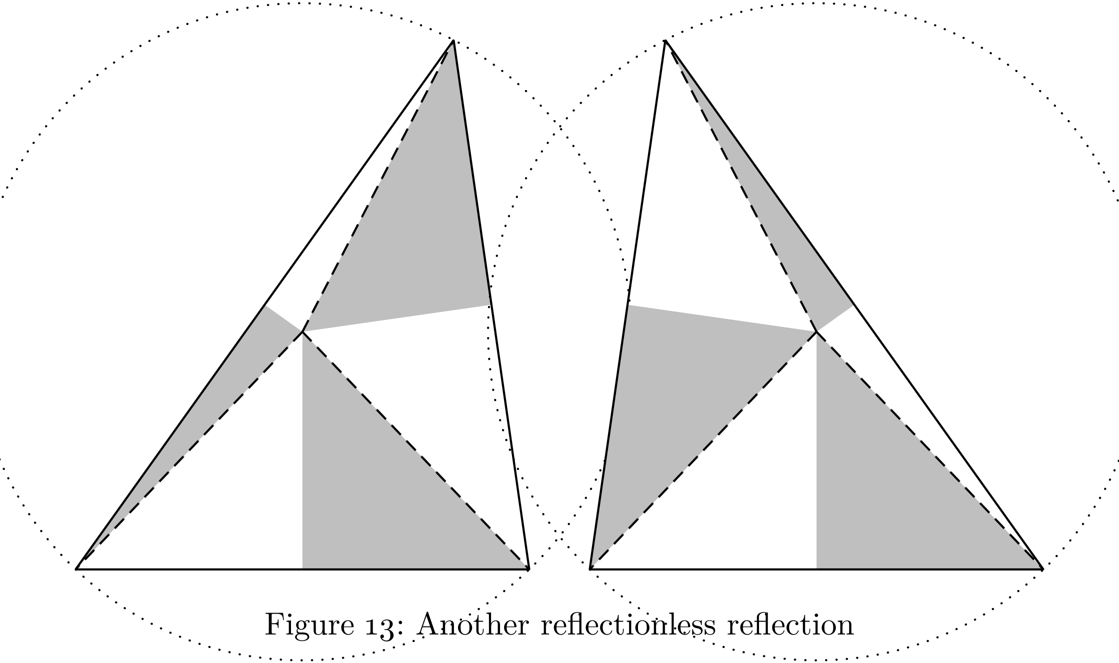

Figure 13: Another reflectionless reflection

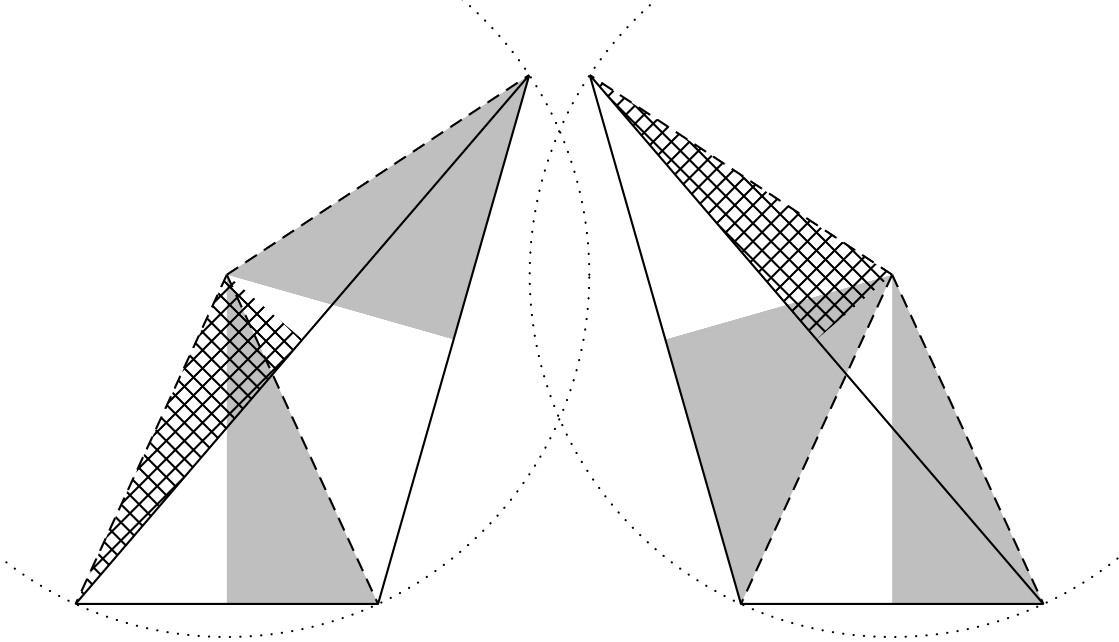

Figure 14: Equicomplementability without reflection

we can get the mirror image of the pyramid on ABC that has apex O by rearranging the pyramids on AKB, BKC, and CKA that have the same apex. By doing the same with the pyramids on the other three faces of the original tetrahedron, $ABCD$, we can obtain *its* mirror image.

As in two dimensions, we may be showing not equidecomposability, but only equicomplementability. For equidecomposability in all cases, we can again work with the incenter, which is now the point of intersection of any three of those six planes, each of which contains one of the edges of the tetrahedron and bisects the *dihedral* angle formed by the sides of the tetrahedron that share that edge. A dihedral angle, or its measurement, is what Euclid calls ἐπιπέδου πρὸς ἐπίπεδον κλίσις, *inclination of a plane to a plane,* in Definition 6 of Book XI.

Apparently Juel made our observation in 1903, writing in Danish [14, p. 55], again interpreted by Google:

> Hertil kan man benytte sig af Centret for den omskrevne Kugle, ifald dette falder indeni Tetraedret, men i alle Tilfælde af Centret O for den indskrevne Kugle. Lad Tetraedret være $ABCD$, og lad os fra O

> fælde de Vinkelrette OO_1, OO_2 og OO' henholdsvis paa Sidefladerne ABC, ABD og paa Kanten AB. Punkterne OO_1O_2O' ligge da i én Plan, og Pyramiden $A - OO_1O'O_2$ er symmetrisk med sig selv for Planen $AO'O$ som Symmetriplan. Men det betragtede Tetraeder kan øjensynlig opfattes som en Sum af 12 fire sidede Pyramider af samme Art, som den nævnte.
>
> For this purpose one can use the center of the circumscribed sphere, if this falls inside the tetrahedron, but in all cases the center O of the inscribed sphere. Let the tetrahedron be $ABCD$, and let us drop from O the perpendiculars OO_1, OO_2 and OO' respectively on the lateral surfaces ABC, ABD and on the edge AB. The points OO_1O_2O' then lie in one plane, and the pyramid $A - OO_1O'O_2$ is symmetrical with itself for the plane $AO'O$ as the plane of symmetry. But the tetrahedron under consideration can obviously be understood as a sum of 12 four-sided pyramids of the same type as the one mentioned.

We can join those four-sided pyramids in pairs. For example, we can match up the base of what Juel calls $A - OO_1O'O_2$ with the base of $B - OO_1O'O_2$. We obtain not an octahedron, but a hexahedron, since

AO_1O' and BO_1O' are the parts of the single triangle ABO_1, as AO_2O' and BO_2O' are the parts of ABO_2. The hexadron has ABO as a plane of symmetry.

There is one hexahedron for each edge of the original tetrahedron. If we reflect each of the six hexahedra about its plane of symmetry, then reflect the whole original tetrahedron, this is tantamount to rearranging the hexahedra without reflection.

That was observed, for a tetrahedron $A_1A_2A_3A_4$, by Jessen in 1968 [13, p. 242]:

> Let O denote the centre of the inscribed sphere and O_i its projection on the face $A_jA_kA_l$ (where i, j, k, and l are distinct). Then P is composed of the six polyhedra $OO_iO_jA_kA_l$ (where again i, j, k, and l are distinct). Since these polyhedra are self-symmetric, they are congruent [without reflection] to the corresponding polyhedra into which [the reflection of the tetrahedron] can be decomposed.

Frederickson reports on all of this [7, p. 231].

Perhaps Jessen did not need to be so clever with notation, although it may be satisfying to know that such notation is *possible.* The point is that we can analyze a tetrahedron into six hexahedra, each formed from a pair of symmetrical tetrahedra sharing a base. That base is the triangle whose base and apex are respectively an edge AB and the incenter O of the original tetrahedron. The apices of the symmetrical tetrahedra sharing the base ABO are the points of tangency of the inscribed sphere with the faces of the original tetrahedron that share the edge AB.

We might construct illustrations as follows. We start with triangles ABC and BCD, as in Figure 15. They can be any triangles whatsoever, as long as they lie on opposite sides of the common base BC. We are going to fold the quadrilateral $ABDC$ along BC so as to make some dihedral angle, of any size whatever. Then D will become the apex of a pyramid on ABC. When the other sides of the pyramid are folded back down, they will make triangles ABF and ACE, as in Figure 16. Here E and F lie where a circle whose center is A meets respectively the circles with centers B and C that pass through D. Thus, the first circle must not be to big or too small to meet the other

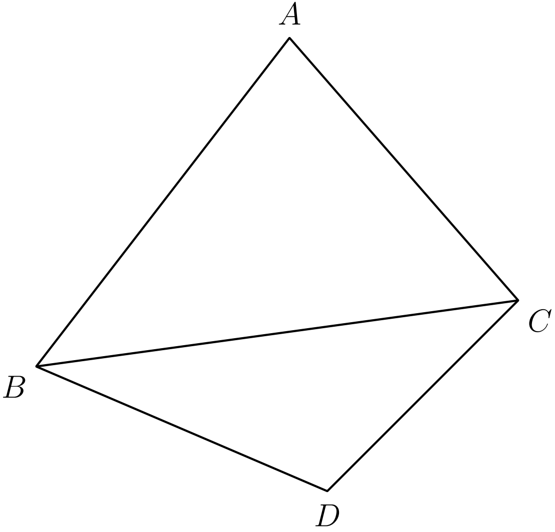

Figure 15: Two triangles

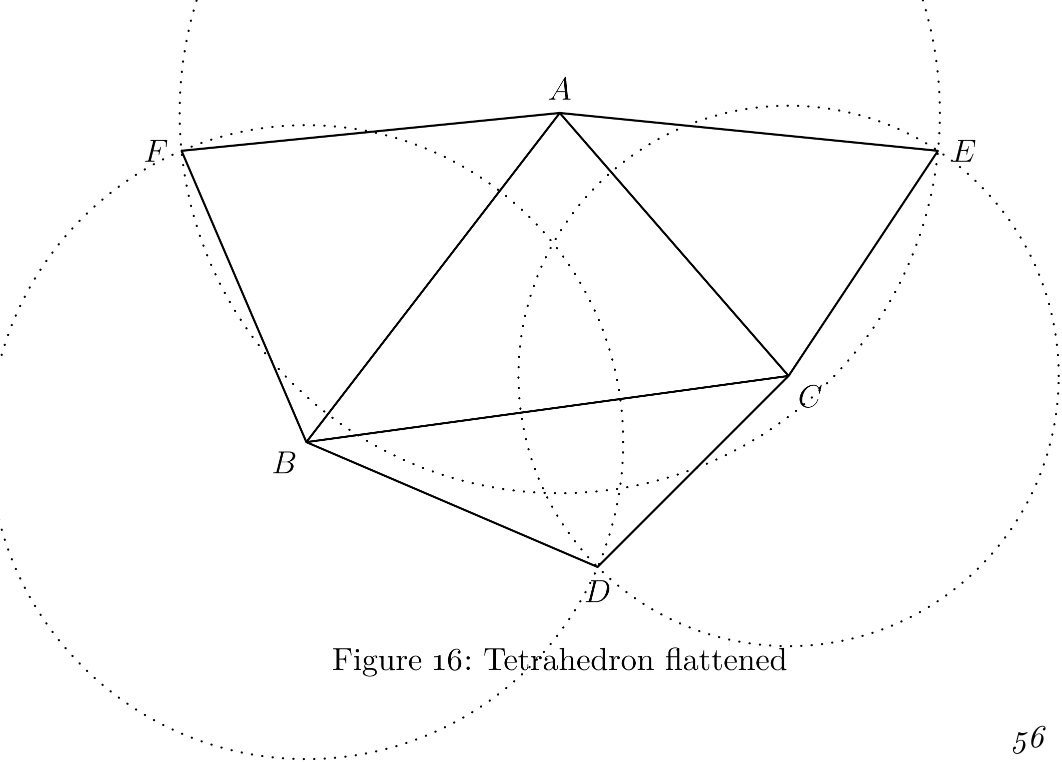

Figure 16: Tetrahedron flattened

two circles. This yields an analogue of *Elements* I.20 and 22, which together are that three lengths are the sides of a triangle if and only if any two together exceed the third.

We now construct the points where the sphere inscribed in our tetrahedron will be tangent to the faces. First we let

$$\angle CAE = \angle XAC, \qquad \frac{1}{2}\angle FAX = \angle FAY, \qquad \angle YAB = \angle FAG,$$

as in Figure 17. Thus

$$\begin{aligned}\angle FAG = \angle YAB &= \angle FAB - \angle FAY \\ &= \angle FAB - \frac{1}{2}\angle FAX \\ &= \angle FAB - \frac{1}{2}(\angle FAB + \angle BAC - \angle XAC) \\ &= \angle FAB - \frac{1}{2}(\angle FAB + \angle BAC - \angle CAE),\end{aligned}$$

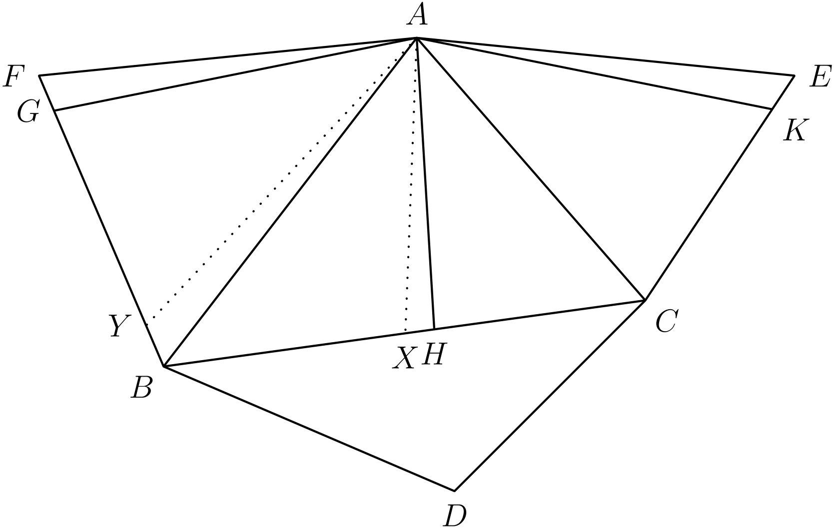

Figure 17: Tetrahedron unfolded

and finally

$$\angle FAG = \frac{1}{2}\left(\angle CAE + \angle FAB - \angle BAC\right).$$

Since $\angle GAB = \angle FAB - \angle FAG$, we can conclude

$$\angle GAB = \frac{1}{2}\left(\angle FAB + \angle BAC - \angle CAE\right).$$

We now let

$$\angle GAB = \angle BAH, \qquad\qquad \angle HAC = \angle CAK.$$

Then

$$\angle HAC = \frac{1}{2}\left(\angle BAC + \angle CAE - \angle FAB\right).$$

It follows then

$$\angle FAG = \angle KAE.$$

Consequently, when the hexagon of the diagram is folded up to make a tetrahedron, with AE coinciding with AF, the inscribed sphere will be tangent to the sides that share the vertex A at points along AG, AH, and AK.

In the same way, we draw the corresponding lines from B, namely BL, BM, and BN, as in Figure 18. When the tetrahedron is formed, so that points D, E, and F coincide, then the inscribed sphere will be tangent to ABF at the intersection C' of AG and BN, and to ABC at the intersection D' of AH and BM. The center of the inscribed sphere being O, the pyramids on the base ABO with apices C' and D' will be reflections of one another. There will be similarly paired pyramids for each of the other five edges of the tetrahedron, and they will fill out the tetrahedron. Meanwhile, AB is the perpendicular bisector of $C'D'$.

5 Hill tetrahedra

We now work towards Dehn's *negative* result. It is a *positive* result, in the sense of showing that Euclid did as well as anybody *can* do, in

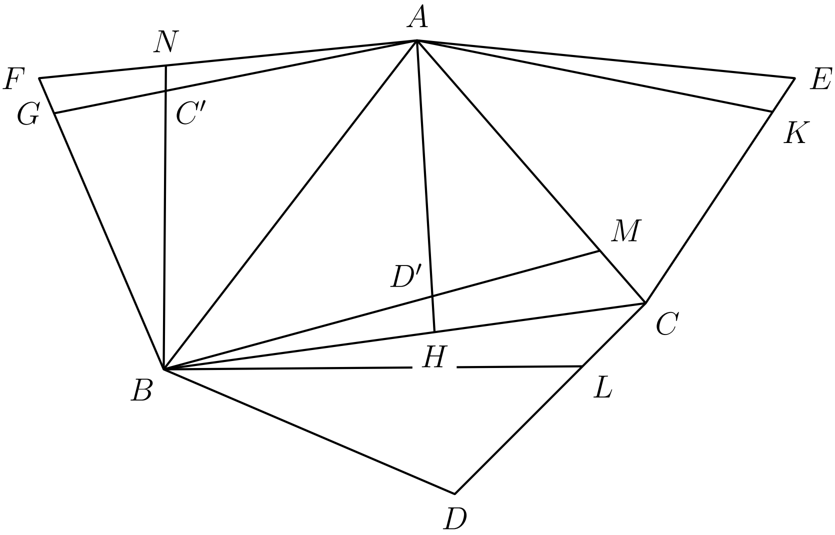

Figure 18: Tetrahedron cut open

showing that pyramids of the same height are as their bases.

We are going to show that, from a cube, we can cut two tetrahedra that are not equicomplementable, even though they are equal, each having a sixth the volume of the cube.

We can cut the cube into six congruent tetrahedra by means of three planes that contain the same diagonal. Indeed, in Figure 19, by *Elements* XI, Proposition 28, the cube GH is bisected by each of the rectangles $GAHD$, $GBHE$, and $GCHF$. See Figure 20 for the view along GH. Three of the tetrahedra are congruent by rotation about GH; for example, so congruent are the tetrahedra that have faces GAB, GCD, and GEF, as in Figure 21. Each of those tetrahedra is also congruent to one of the three other ones by reflection.

As a possible aid (or possibly not) to one's spacial intuition, the faces shown in color in Figure 21 are laid down step by step in Figures 22 and 23, where are colored, in sequence,

1) the rear faces of the cube;
2) half of each of two of the rectangles;

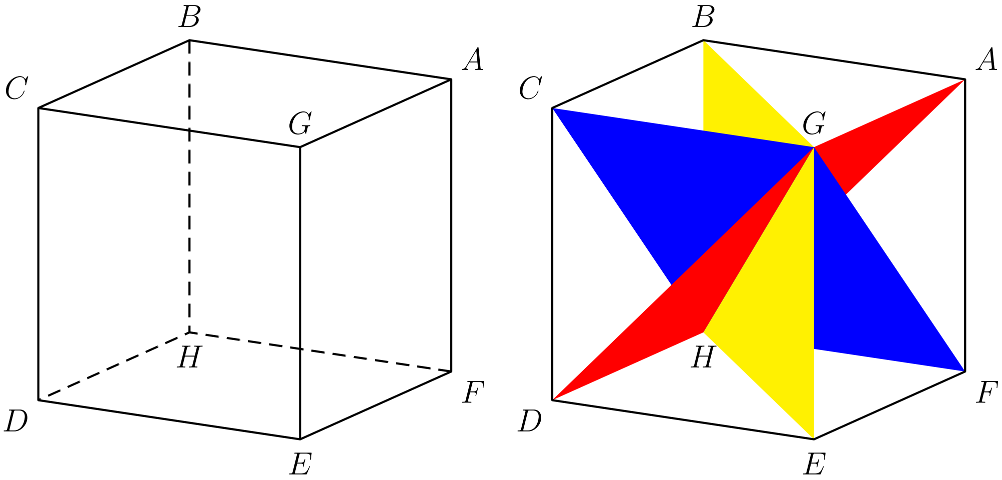

Figure 19: Cube: plain and bisected three ways

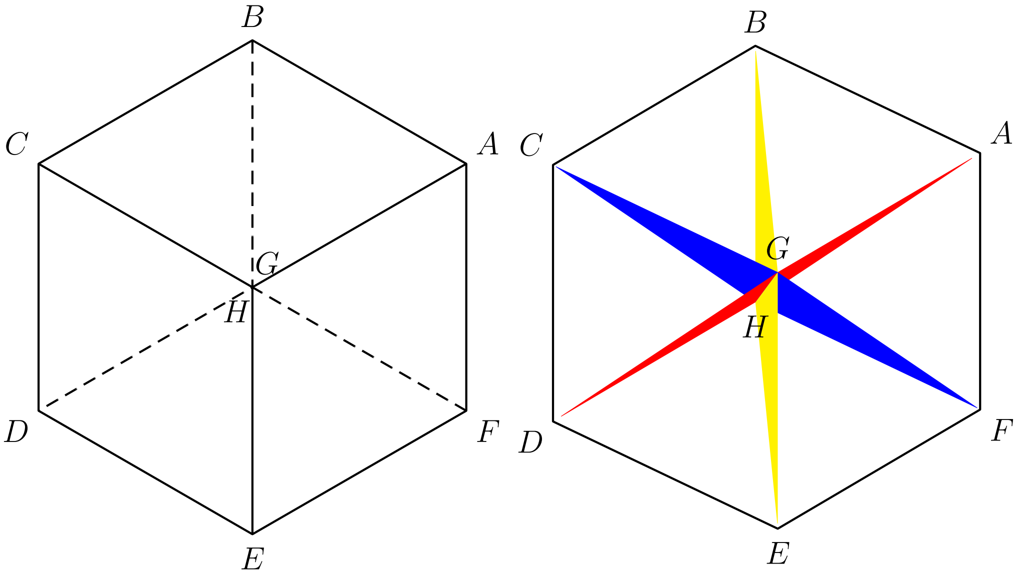

Figure 20: Cube bisected three ways, diagonal view

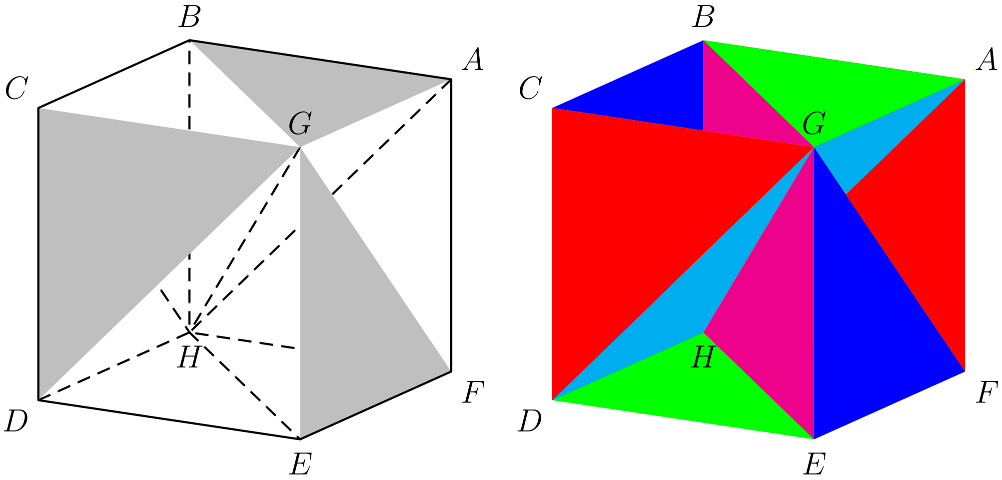

Figure 21: Rotational symmetry

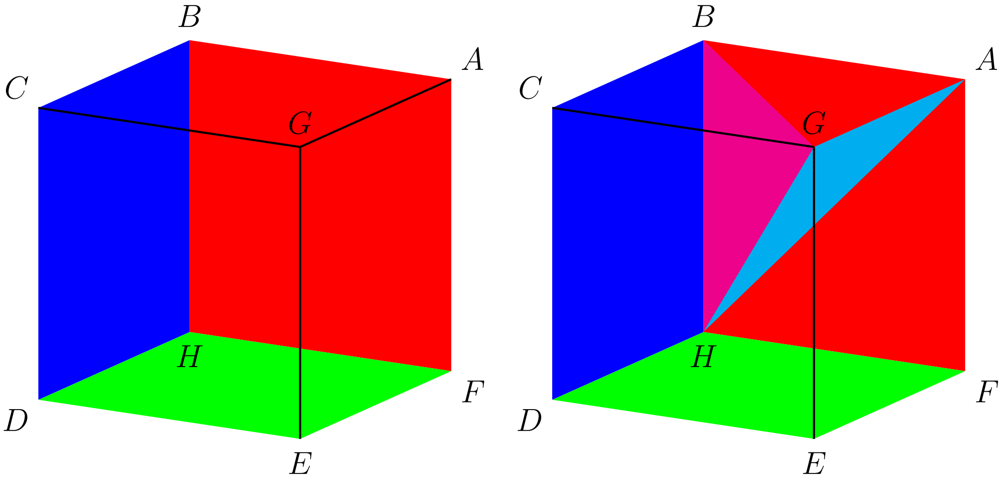

Figure 22: Rotational symmetry in color, steps 1 and 2

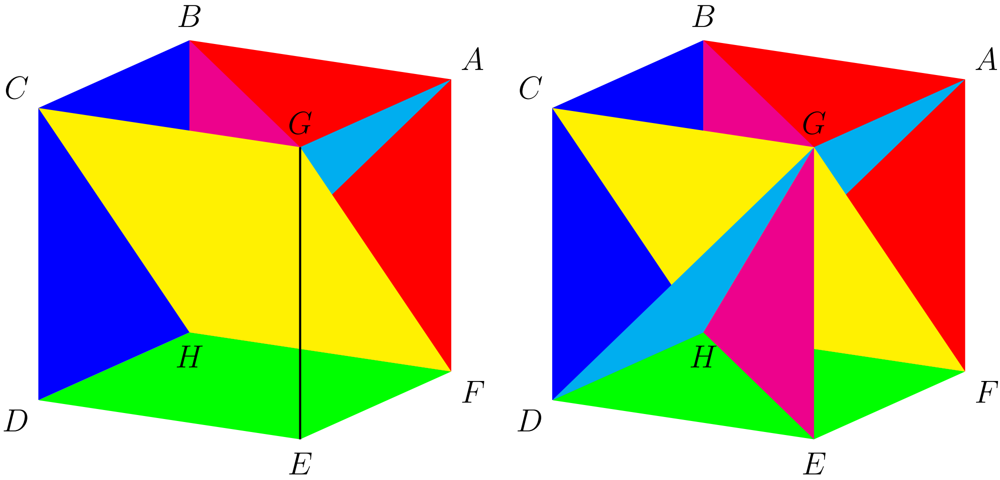

Figure 23: Rotational symmetry in color, steps 3 and 4

3) the third rectangle;

4) the other halves of the former two rectangles.

As the the view along GH in Figure 20 suggests, if we *scale* our figure in that direction, whether by expanding or contracting, we still have an equilateral parallelepiped, which features in Elements XI.36 and can be called a rhombohedron. It is still divided into six congruent tetrahedra.

Those are *Hill tetrahedra* (you can look them up on Wikipedia). They were identified by Hill in 1895 [11], precisely because, as we have just seen, without recourse to *Elements* XII.5, we can know each of them as being a third of a prism with the same base and height. (Strictly, Hill refers explicitly only to XII.3 among Euclid's propositions.)

That a Hill tetrahedron is equidecomposable with a prism on the same base and with a third the height was shown explicitly by Sydler in 1956 [21], as I learned from Frederickson [7, pp. 232f.]. In Figure 24, the tetrahedron is cut by a plane parallel to the base, one-third of

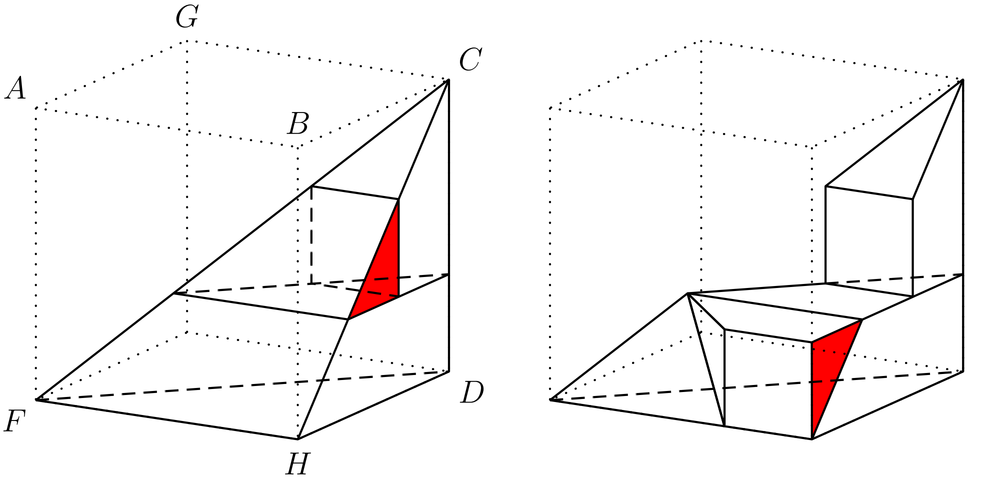

Figure 24: Tetrahedron re-arranged, steps 1 and 2

the way up, and then the similar tetrahedron that has been cut off is cut into congruent parts by a vertical plane. One of those parts swings around, so as to start filling out the bottom part of the tetrahedron into a prism. In Figure 25, the other part of the smaller tetrahedron is divided and re-arranged to fill out the rest of the prism. There is another view in Figure 26.

In the tetrahedron that we have just shown to be equidecomposable with a prism,

- three edges are edges of the cube it was cut from;
- two more edges are diagonals of faces of the cube;
- the last edge is a diagonal of the cube;
- the dihedral angle
  - at one of the short edges and at each of the two mid-length edges is right;
  - at each of the other two short edges is half a right angle;
  - at the long edge is a third of two right angles.

In short, each dihedral angle of our tetrahedron is a half, a third, or a

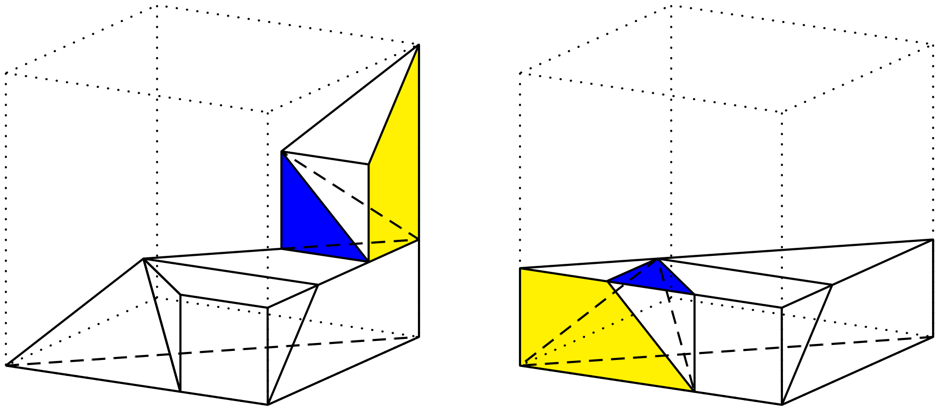

Figure 25: Tetrahedron re-arranged, steps 3 and 4

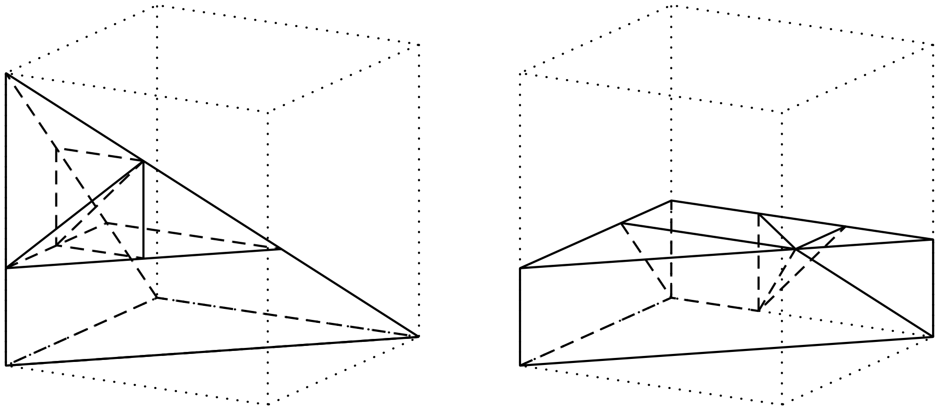

Figure 26: Tetrahedron re-arranged, alternative view

fourth of two right angles. This is why the tetrahedron can be cut up and re-arranged into a prism on an isosceles right triangle. At least, the commensurability of the angles with a right angle will be seen to be a *necessary* condition for what we have done.

6 Equilateral pyramids

We can also cut up the cube in another way, as in Figures 27 and 28, where four congruent tetrahedra have as bases the triangles BFG, BDG, DFG, and BDF respectively, and those triangles are the faces of a fifth tetrahedron. That fifth tetrahedron is double each of the other four, because each of those four is equal to each of the six earlier tetrahedra. For example, the tetrahedra $BDFH$ and $CDFH$, as in Figure 29, are equal, because they have congruent bases and the same height. However, the two tetrahedra are not going to be equicomplementable.

We can count the tetrahedron $BDFH$ as an *equilateral pyramid* with

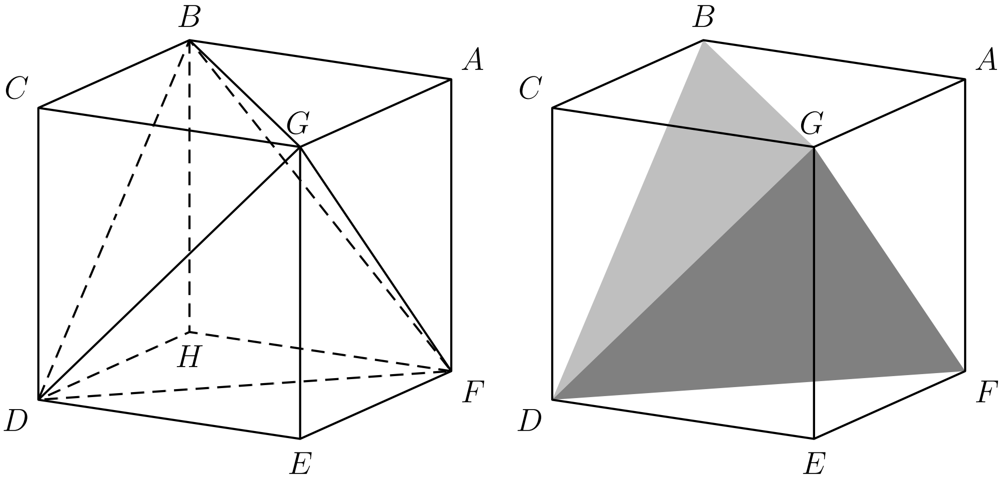

Figure 27: Cube with four tetrahedra cut off, one remaining

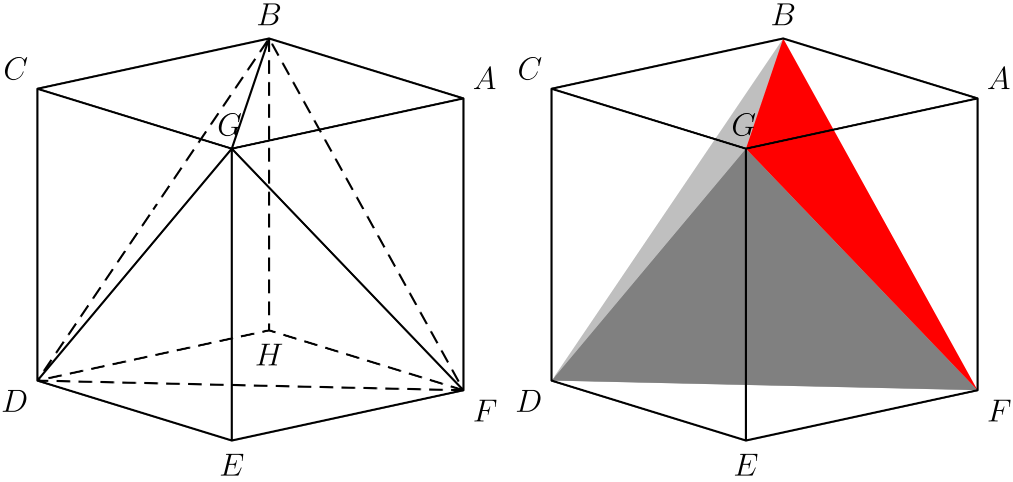

Figure 28: Same cube, rotated

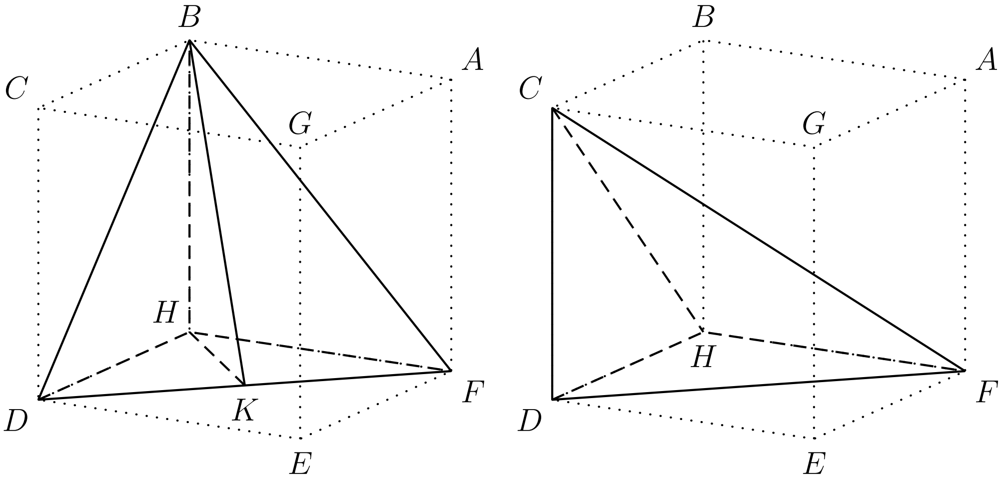

Figure 29: Equal incongruent tetrahedra

base BDF, because the edges joining the vertices of the base to the apex are all equal to one another. The base is therefore equilateral as well. Indeed, of the pyramid,

- the edges meeting the apex are edges of the cube;
- the edges of the base are diagonals of faces of the cube;
- the dihedral angle at each of the shorter edges is right;
- the dihedral angle at each of the longer edges has $\sqrt{3}$ as its *secant*, thus $1/\sqrt{3}$ as its *cosine*.

For example, if the perpendicular from B to DF has the foot K, then also HK is at right angles to DF, so the dihedral angle of the tetrahedron at DF has the measure of BKH. If $BH = 1$, as on the left of Figure 30, where we are looking at the cube along DF, then

$$HK = \frac{1}{2}HE = \frac{\sqrt{2}}{2}, \qquad BK^2 = BH^2 + HK^2 = 1 + \frac{1}{2} = \frac{3}{2},$$

so

$$\frac{BK}{HK} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{3}.$$

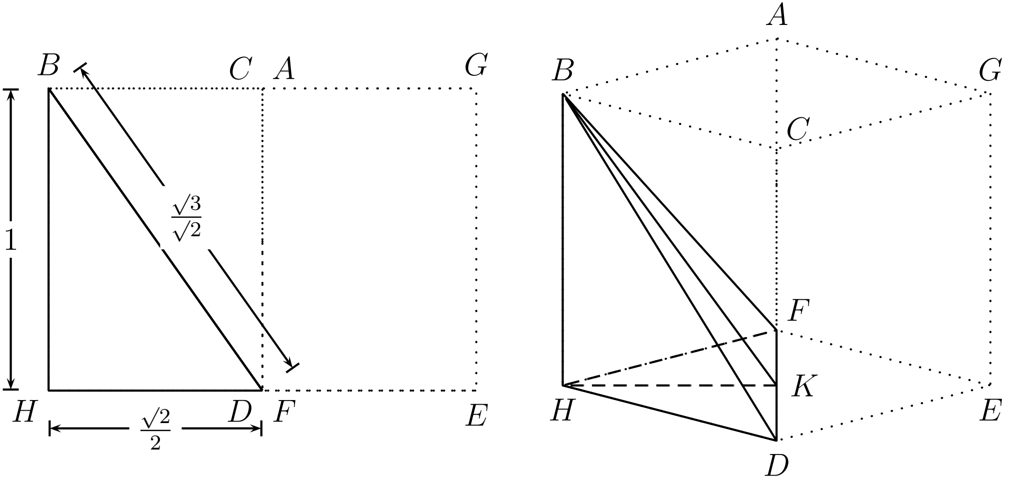

Figure 30: Trigonometry

We show now that the angle whose secant is $\sqrt{3}$ is incommensurable with a right angle. This will mean that the equilateral pyramid of Figure 29 is not equidecomposable or even equicomplementable with the Hill tetrahedron there, basically because there is no way to get rid of the angle arcsec $\sqrt{3}$.

7 Trigonometry

We shall work that out in more detail in § 8. Meanwhile, we are going to continue doing trigonomety, taking the general idea from the chapter "Some Irrational Numbers" in *Proofs from the Book* [1, pp. 52f.] (which book also has a chapter on Hilbert's Third Problem).

We are going to start with **Ptolemy's Theorem,** from the *Almagest* [18, pages 16–7], [19, pages 50–1], [22, pages 422–5]:

Ἔστω γὰρ	For, be

κύκλος ἐγγεγραμμένον	inscribed in a circle,
ἔχων τετράπλευρον	having four sides,
τυχὸν	at random,
τὸ ΑΒΓΔ,	$AB\Gamma\Delta$,
καὶ ἐπεζεύχθωσαν	and have been joined
αἱ ΑΓ καὶ ΒΔ.	$A\Gamma$ and $B\Delta$.
δεικτέον, ὅτι	To be shown is that
τὸ ὑπὸ τῶν ΑΓ καὶ ΒΔ	the by-$A\Gamma$-and-$B\Delta$
περιεχόμενον ὀρθογώνιον	contained rectangle
ἴσον ἐστὶ συναμφοτέροις	is equal to, together,
τῷ τε ὑπὸ τῶν ΑΒ, ΔΓ	$AB \cdot \Delta\Gamma$
καὶ τῷ ὑπὸ τῶν ΑΔ, ΒΓ.	and $A\Delta \cdot B\Gamma$.

$AB\Gamma\Delta$ being as in Figure 31, by *Elements* III, Proposition 26,

$$\angle A\Gamma B = \angle A\Delta B.$$

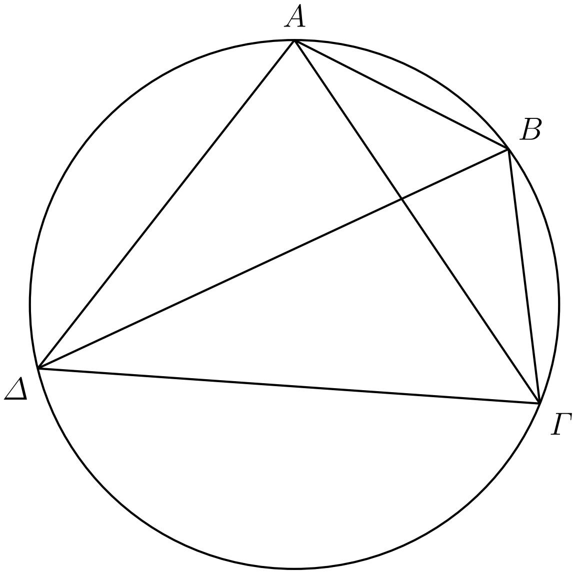

Figure 31: Ptolemy's Theorem: $A\Gamma \cdot B\Delta = AB \cdot \Gamma\Delta + B\Gamma \cdot A\Delta$

We let E on $B\Delta$ ensure

$$\angle \Gamma AB = \angle \Delta AE,$$

as in Figure 32. We now have two pairs of similar triangles, by *Elements* VI, Proposition 4. We apply to them VI.16. Thus, we have

$$\begin{aligned} \Gamma AB &\sim \Delta AE, & \Gamma\Delta A &\sim BEA, \\ A\Gamma : B\Gamma &:: A\Delta : E\Delta, & A\Gamma : \Gamma\Delta &:: AB : BE, \\ A\Gamma \cdot E\Delta &= A\Delta \cdot B\Gamma, & A\Gamma \cdot BE &= AB \cdot \Gamma\Delta. \end{aligned}$$

By Proposition II.1,

$$A\Gamma \cdot B\Delta = A\Gamma \cdot (BE + E\Delta) = A\Gamma \cdot BE + A\Gamma \cdot E\Delta.$$

All this combined gives

$$A\Gamma \cdot B\Delta = AB \cdot \Gamma\Delta + B\Gamma \cdot A\Delta$$

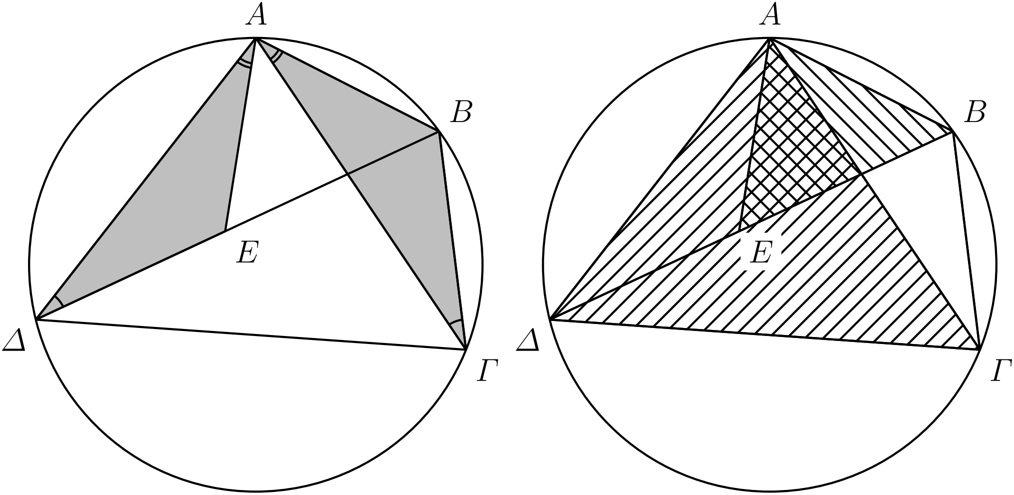

Figure 32: Proof of Ptolemy's Theorem

– the product of the diagonals is the sum of the products of the opposite sides.

Our quadrilateral with diagonals drawn is a *complete quadrangle*: it is a quadrangle, in the sense of having four vertices where angles *can* be drawn, and it is complete in the sense that for every pair of vertices, there is an edge joining them.

As Ptolemy does, we consider the special case where one of the sides of the complete quadrangle is a diameter of the circumscribed circle. If that diameter has unit length, then, in modern notation, the other sides are as in Figures 33 and 34. Thus, we have the angle addition and subtraction formulas from high-school algebra II [23, pp. 540f],

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

By addition, we have a product of cosines as a sum:

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta.$$

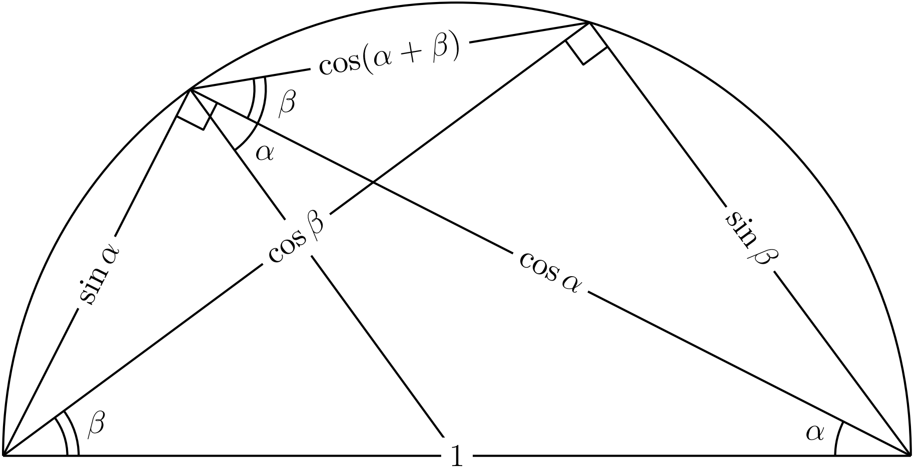

Figure 33: Angle addition

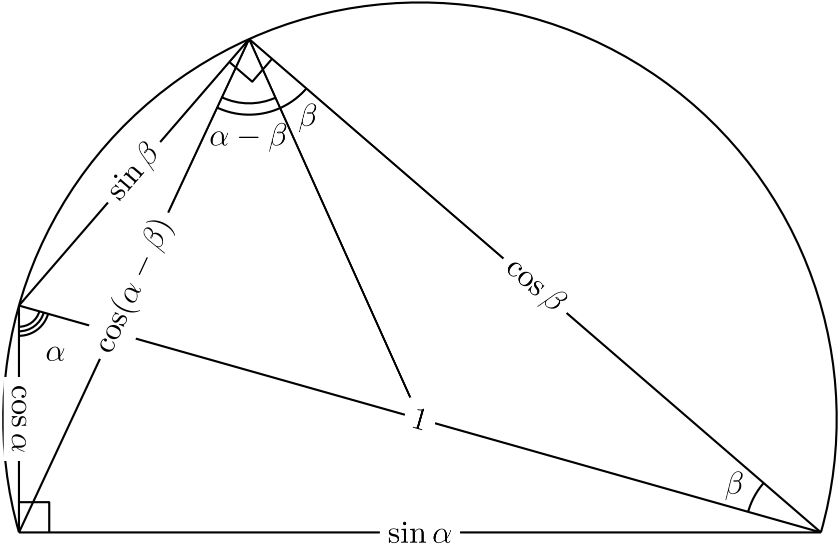

Figure 34: Angle subtraction

Letting

$$\alpha + \beta = \varphi, \qquad \alpha - \beta = \psi, \qquad 2\alpha = \varphi + \psi, \qquad 2\beta = \varphi - \psi,$$

we have a sum of cosines as a product:

$$\cos\varphi + \cos\psi = 2\cos\frac{\varphi+\psi}{2}\cos\frac{\varphi-\psi}{2}.$$

Letting

$$\varphi = (k+1)\vartheta, \qquad \psi = (k-1)\vartheta, \qquad \frac{\varphi+\psi}{2} = k\vartheta, \qquad \frac{\varphi-\psi}{2} = \vartheta,$$

we obtain

$$\cos\bigl((k+1)\vartheta\bigr) + \cos\bigl((k-1)\vartheta\bigr) = 2\cos(k\vartheta)\cos\vartheta$$

or rather

$$\cos\bigl((k+1)\vartheta\bigr) = 2\cos(k\vartheta)\cos\vartheta - \cos\bigl((k-1)\vartheta\bigr).$$

Table 2: Multiple angle formulas

$$
\begin{aligned}
\cos 0 &= 1 \\
\cos\vartheta &= \cos\vartheta \\
\cos(2\vartheta) &= 2\cos^2\vartheta - 1 \\
\cos(3\vartheta) &= 4\cos^3\vartheta - 3\cos\vartheta \\
\cos(4\vartheta) &= 8\cos^4\vartheta - 8\cos^2\vartheta + 1 \\
\cos(5\vartheta) &= 16\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta \\
\cos(6\vartheta) &= 32\cos^6\vartheta - 48\cos^4\vartheta + 18\cos^2\vartheta - 1 \\
\cos(7\vartheta) &= 64\cos^7\vartheta - 112\cos^5\vartheta + 56\cos^3\vartheta - 7\cos\vartheta \\
\cos(8\vartheta) &= 128\cos^8\vartheta - 256\cos^6\vartheta + 160\cos^4\vartheta - 32\cos^2\vartheta + 1 \\
\cos(9\vartheta) &= 256\cos^9\vartheta - 576\cos^7\vartheta + 432\cos^5\vartheta - 120\cos^3\vartheta + 9\cos\vartheta
\end{aligned}
$$

Table 3: Coefficients in multiple angle formulas

									1				
								1		−1			
							2		−3		1		
						4		−8		5		−1	
					8		−20		18		−7		1
				16		−48		56		−32		9	
			32		−112		160		−120				
		64		−256		432							
	128		−576										
256													

With this, we can work out as many multiple angle formulas as we like, as in Table 2. We can compute coefficients mechanically, in a kind of analogue of Pascal's Triangle, as in Table 3, where each entry but the first two (counting from the top, then left) is twice the one (if there is one) just above on the right, minus the one (if there is one) just above on the left. One can prove for example that the sum of the entries of each row but the first is zero. This could be a way to check one's computations.

What is important for us is that $\cos(k\vartheta)$ is a sum of *integral* multiples – alternatively, a difference of sums of *numerical* multiples – of powers of $\cos\vartheta$, up to the power k. For example,

$$\cos(6\vartheta) = 32\cos^6\vartheta - 48\cos^4\vartheta + 18\cos^2\vartheta - 1,$$

and 32, -48, 18, and -1 are integers; also

$$\cos(6\vartheta) = 32\cos^6\vartheta + 18\cos^2\vartheta - (48\cos^4\vartheta + 1).$$

Moreover, the powers of $\cos\vartheta$ always have only *even* exponents, when k is even. Finally, the coefficient on $\cos^k\vartheta$ is 2^{k-1}. Thus, for some integers a_i (that is, for some integers a_1, a_2, . . . , a_{k-1}, and a_k),

$$\begin{aligned}\cos\left(2k\arccos\frac{1}{\sqrt{3}}\right) &= \frac{2^{2k-1}}{3^k}+\frac{a_1}{3^{k-1}}+\frac{a_2}{3^{k-2}}+\dots+\frac{a_{k-1}}{3}+a_k\\ &= \frac{2^{2k-1}+3a_1+9a_2+\dots+3^{k-1}a_{k-1}+3^k a_k}{3^k}\\ &= \frac{2^{2k-1}+3(a_1+3a_2+\dots+3^{k-2}a_{k-1}+3^{k-1}a_k)}{3^k}.\end{aligned}$$

This is never ± 1, since the numerator is never a multiple of 3.

The measure of two right angles is π. Since the cosine of a multiple of π is ± 1, no *even* multiple of $\arccos(1/\sqrt{3})$ is ever a multiple of π. Therefore no multiple at all is.

Another way to express that conclusion is that $\arccos(1/\sqrt{3})$ and π are *incommensurable*.

By the definition in *Elements* X, magnitudes are *commensurable* when they have the same measure. By the definition in Book V, we can call that measure a *part*. If magnitudes a and b have the common part c, then we can let d be the same multiple of a that b is of c. This means

$$d : a :: b : c.$$

By alternation, as in Proposition V.16,

$$d : b :: a : c,$$

and in particular, d is the same multiple of b that a is of c. Thus, d is a common multiple of a and b.

Suppose conversely that a and b have a common multiple. Then

$$ay = bx$$

for some numbers y and x. These are numbers in the sense of *Elements* VII: multitudes of units. Then for example ay means a sum of as many copies of a as there are units in y. We do allow a unit by itself to count as a number. From our equation then, by the definition of proportion in Book V,

$$a : b :: x : y.$$

Indeed, for any numbers z and w, by Proposition VII.16 (commutativity of multiplication of numbers),

$$(az)(yw) = ayzw = bxzw = (bz)(xw),$$

so

$$\begin{aligned} az < bz &\iff yw > xw, \\ az = bz &\iff yw = xw, \\ az > bz &\iff yw < xw. \end{aligned}$$

Thus a and b are in the ratio of numbers, so they are commensurable by *Elements* X.6.

We have now shown that having a common multiple is the same as being commensurable; having no common multiple, incommensurable.

8 Wedges

In expressing the theorem named for him and worked out in the previous section, Ptolemy talked about rectangles contained by certain lengths; *we* ended up talking about those rectangles as the *products* of those lengths. We may likewise think of a rectangular parallelepiped as the product of its base and its height.

We are going to consider the product of the length and the dihedral angle of the edge of a polyhedron. We can understand this angle in isolation as a part of a plane cut off by two *rays* having a common endpoint. It is like a triangle with a fixed apex, and a fixed angle there, but we don't care how far away the base is. With two such infinite triangles that are congruent as opposite **bases,** we can form a right prism whose height is the length of the edge of the polyhedron. Let us call this infinite prism a **wedge.** The straight line joining the vertices of the two bases of the wedge is its **edge.** Two infinite rectangular **faces** of the wedge meet at the edge.

I don't know whether anybody else defines a wedge geometrically in that way. A wedge for splitting a log has an edge, which is placed against the log. The faces of the wedge slide against the wood as it is exposed by the splitting. I have found no scholarly suggestion that "edge" and "wedge" are cognate, even though they go back to Old English as *ecg* and *wecg*.

A log-splitting wedge does also have a face that it is struck on. We may consider each of *our* wedges as having such face, which is part of the surface of a cylinder. This seems to be what Dehn does. In his paper [3, p. 470], as shown in our Figure 35, Dehn's Figure 7 would seem to be a diagram of a wedge, cut off in what is called a **Cylinderplatte**:

> Diese Verhältnisse wollen wir uns folgendermassen veranschaulichen: Wir bezeichnen ein Stück der Ebene, das durch eine Strecke PQ und zwei (Halb)Strahlen der Ebene, dei in P und Q auf PQ senkrecht und nach derselben Seite von PQ gerichtet sind, begrenzt wird, als einen von PQ ausgehenden Halbstreifen. Jeden zu einer Kante t_i des Kantenzuges AB gehörigen Flächenwinkel τ_i begrenzen wir durch zwei von t_i ausgehende Halbstreifen (Fig. 7). Sodann beschreiben wir um AB als Axe einen geraden Kreiscylinder mit dem Radius 1. Infolge des

die zu den Kanten des Kantenzuges AB gehören, sich an jeder Stelle von AB zu $2R$, x_i oder $4R$ ergänzen, je nachdem einer der oben aufgezählten Fälle vorliegt.

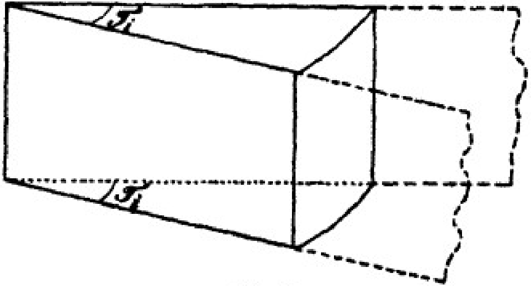

Fig. 7.

Diese Verhältnisse wollen wir uns folgendermassen veranschaulichen: Wir bezeichnen ein Stück der Ebene, das durch eine Strecke PQ und zwei (Halb)Strahlen der Ebene, die in P und Q auf PQ senkrecht und nach derselben Seite von PQ gerichtet sind, begrenzt wird, als einen von PQ ausgehenden Halbstreifen. Jeden zu einer Kante t_i des Kantenzuges AB gehörigen Flächenwinkel τ_i begrenzen wir durch zwei von t_i ausgehende Halbstreifen (Fig. 7). Sodann beschreiben wir um AB als

Figure 35: Dehn's "cylindrical plate"

> Schnittes der Halbstreifen mit dieser Cylinderfläche wird eine Kante und der zugehörige Flächenwinkel stets repräsentirt durch ein Stück der Cylinderfläche, das wir eine rechtwinklige *Cylinderplatte* nennen wollen.
>
> We will illustrate these relationships as follows: We designate a segment of the plane bounded by a line segment PQ and two (half) rays of the plane, perpendicular to PQ at P and Q and directed towards the same side of PQ, as a semistrip extending from PQ. We delimit each surface angle τ_i belonging to an edge t_i of the edge chain AB by two strips extending from t_i (Fig. 7). Then, centered on AB as its axis, we describe a right circular cylinder with radius 1. As a result of the intersection of the semistrips with this cylindrical surface, an edge and the corresponding surface angle is always represented by a segment of the cylindrical surface, which we will call a right-angled *cylindrical plate.*

Let us just talk about wedges.

As he sets out on his own exposition of Dehn's solution of Hilbert's Third Problem, Boltianskii [2, p. 100] says (with references omitted),

> Dehn's own exposition was hard to understand. In 1903 Kagan published a paper in which Dehn's argument was considerably refined, and presented in a more systematic and readable fashion. This was, so to speak, the rebirth of Dehn's paper. In the 1950's a number of interesting results in the theory of equidecomposability were obtained by the Swiss geometer Hadwiger and his students. Their work allows one to take a new look at the work of Dehn, and to obtain Dehn's basic result by using transparent ideas in a modern treatment.

A "modern treatment" has its own difficulties. I am trying to base my own treatment as much as possible on Euclid.

By its length and its dihedral angle then, each edge of a polyhedron determines a wedge. The collection of all of these wedges is the **Dehn invariant** of the solid in question. When we understand this collection properly, then the Dehn invariant will indeed be *invariant* in the sense that, if we cut up the polyhedron into smaller polyhedra, the sum of all of *their* Dehn invariants will be just the Dehn invariant of the original polyhedron.

There are three ways to make a collection of wedges smaller in the sense of being less numerous.

1. If two wedges have equal angles and thus congruent bases, then we can glue the two wedges together at bases.
2. If two wedges have equal edges and thus congruent faces, then we can glue the two wedges together at faces.
3. If two wedges have equal edges and *supplementary* angles, meaning the angles are equal to two right angles, then the wedges cancel or annihilate one another.

We can also reverse any of these three actions.

Symbolically, if the edge of a wedge measures ℓ, and the angle measures α, we shall write the measure of the wedge as a whole as

$$\ell \otimes \alpha.$$

The new symbol $\otimes$ serves, first of all, to separate completely different kinds of magnitudes.

Lengths and areas are different kinds of magnitudes, but not *completely* different. Given two lengths and an area bounded by *straight* lines, we can construct a new area whose ratio to the given one is the ratio of the two lengths. In Book XII, Proposition 2, which is that circles are as the squares on their bases, Euclid even assumes (though he does not strictly need to) that the one *circle* is to *some* area as the one square is to the other.

We cannot always do the same thing with two lengths and an angle. The notorious example would seem to be that we cannot *trisect* an angle. Indeed, from Table 2 we have

$$\cos(3\vartheta) = 4\cos^3\vartheta - 3\cos\vartheta,$$

which is a cubic equation in $\cos\vartheta$ when we treat $\cos(3\vartheta)$ as given. With ruler and compass, we can solve only quadratic equations.

Nonetheless, we *are* going to assume that, like every length, every angle has every *part* in the sense of *Elements* Book V: a half, a third, a fourth, and so on. However, given *incommensurable* lengths ℓ and m, along with an angle α, we shall not assume that the proportion

$$\ell : m :: \alpha : \xi$$

is soluble.

We can write our first two rules for combining wedges as

$$\ell \otimes \alpha + m \otimes \alpha = (\ell + m) \otimes \alpha,$$
$$\ell \otimes \alpha + \ell \otimes \beta = \ell \otimes (\alpha + \beta).$$

In short, the "multiplication" indicated by the symbol $\otimes$ distributes over addition on either side.

If two wedges are congruent, then we have a choice of how to write their sum:

$$2\ell \otimes \alpha = \ell \otimes \alpha + \ell \otimes \alpha = \ell \otimes 2\alpha.$$

In the same way,

$$3\ell \otimes \alpha = \ell \otimes 3\alpha,$$
$$4\ell \otimes \alpha = \ell \otimes 4\alpha,$$

and so on.

Our third rule of combining wedges is

$$\ell \otimes \alpha + \ell \otimes (\pi - \alpha) = 0,$$

where again π is the measure of two right angles. Given the first two rules, we can write the third rule as

$$\ell \otimes \pi = 0.$$

This in particular is what will make the Dehn invariant invariant in the sense described.

A polyhedron has faces that are polygons. These meet along the edges of the polyhedron, and the edges meet at vertices.

Along an edge of a polyhedron, we may designate a point as a new vertex, thus dividing the one edge into two. The Dehn invariant of the polyhedron is unchanged.

Neither is it changed if we cut up a face into several polygons; for, in this case, the wedge associated with each new edge has an angle equal to two right angles, and this counts as nothing, by our definition.

We could just draw a new edge part of the way into a face. Perhaps then we should count the face as having *two* new edges, but it doesn't matter: their wedges still count as nothing for the Dehn invariant.

If we cut into the polyhedron, as it were with a knife with a straight edge, this gives the polyhedron two new faces, which meet one another. The new edge where the knife stops has a dihedral angle whose measure is four right angles, or 2π, so again the new wedge counts for nothing.

If the knife cuts into the polyhedron along an edge, this effectively cuts the wedge of that edge into two, whose sum is the original edge.

Dividing a polyhedron into smaller polyhedra may involve cutting in more subtle ways. Still, every new edge will be part of an existing edge, or will lie along one of the existing faces, or will penetrate the interior of the polyhedron. In each case, the Dehn invariant is unchanged.

Is that a proof? Have we overlooked some possibility? Not to my knowledge – but that observation, in itself, is not a proof!

If the length of a side of our cube is 1, then, by our earlier analysis in § 5, the Dehn invariant of the latter tetrahedron – the Hill tetrahedron

– of Figure 29 on page 76 is

$$(1+2\sqrt{2}) \otimes \frac{\pi}{2} + 2 \otimes \frac{\pi}{4} + \sqrt{3} \otimes \frac{\pi}{3}.$$

Since

$$2 \otimes \frac{\pi}{4} = 1 \otimes 2 \cdot \frac{\pi}{4} = 1 \otimes \frac{\pi}{2},$$

the Dehn invariant simplifies to

$$(2+2\sqrt{2}) \otimes \frac{\pi}{2} + \sqrt{3} \otimes \frac{\pi}{3}.$$

But then, we didn't really need to do this, since always

$$\ell \otimes \frac{\pi}{2} = \frac{\ell}{2} \otimes \pi = 0, \qquad \ell \otimes \frac{\pi}{3} = \frac{\ell}{3} \otimes \pi = 0.$$

Thus our Dehn invariant reduces to 0.

By the work of § 7, the Dehn invariant of the former tetrahedron – the equilateral pyramid – of Figure 29 is

$$3 \otimes \frac{\pi}{2} + 3\sqrt{2} \otimes \operatorname{arcsec} \sqrt{3}.$$

This simplifies to

$$3\sqrt{2} \otimes \operatorname{arcsec} \sqrt{3}.$$

We have seen in § 7 that no multiple of $\operatorname{arcsec} \sqrt{3}$ is ever a multiple of π. *If* we can conclude that the Dehn invariant of our equilateral pyramid is nonzero, then the pyramid is not equidecomposable with our Hill tetrahedron, because the Hill tetrahedron has zero Dehn invariant, even though the two polyhedra are equal.

Can we so conclude? We can, and we can conclude even that two tetrahedra are not even equicomplementable. The latter is by the third rule for combining wedges. That rule gives each wedge, and therefore each Dehn invariant, an *inverse.*

If polyhedra that have Dehn invariants x and y respectively are equidecomposable, then we know

$$x = y.$$

If the polyhedra are equicomplementable, then for some Dehn invariant z,

$$x + z = y + z.$$

In that case, we can add the inverse of z, reducing to the previous equation.

We have generally that if ℓ is a length, and α is an angle, then the product

$$\ell \otimes \alpha$$

is nil, provided α is *commensurable* with π.

9 Rational independence

We shall show that the foregoing condition is not only sufficient, but also necessary: if α is incommensurable with π, then $\ell \otimes \alpha \neq 0$.

We shall do this by broadening the scope of the notions of commensurability and incommensurability.

We observed in § 7 that two magnitudes a and b (implicitly of the same kind – for example, both lengths or both angles) are commensurable if and only if the equation

$$ay = bx$$

has a numerical solution. Some number of magnitudes of the same kind are **rationally dependent** (you can look up the term on Wikipedia) if there is a numerical solution to an equation

$$a_1 y_1 + \cdots + a_m y_m = b_1 x_1 + \cdots + b_n x_n,$$

where $a_1, \ldots, a_m$ are some of the magnitudes, and $b_1, \ldots, b_n$ are some of the *rest* of the magnitudes. Alternatively, if the full list of the given magnitudes is $c_1, \ldots, c_r$, then the equation

$$c_1 z_1 + \cdots + c_r z_r = 0$$

should have an *integral* solution that is *not trivial* (meaning not all of the integers z_i should be 0).

If A is a set of magnitudes, and b is a magnitude, then the set whose elements are b and the elements of A is $A \cup \{b\}$. If this is rationally dependent, but A is not, then b is said to be rationally dependent on A.

From any set of magnitudes of the same kind, we can select a maximally independent subset, namely an independent subset on which every other element of the set depends. We can do this by making a list of the magnitudes and consider them one by one for selection. We select each magnitude that is not dependent on the ones already selected.

Suppose the original set of magnitudes is A, and the maximally independent subset is $\{b_1, \ldots, b_n\}$. Then every element of A is of the form

$$\frac{1}{z}\left(b_1 x_1 + \cdots + b_n x_n\right),$$

where z is a number and the x_i are integers. The expression is unique in the following sense. If

$$\frac{1}{z}\left(b_1x_1+\cdots+b_nx_n\right)=\frac{1}{w}\left(b_1y_1+\cdots+b_ny_n\right),$$

where z and w are numbers, and all of the x_i and y_i are integers, then

$$b_1(wx_1-zy_1)+\cdots+b_n(wx_n-zy_n)=0.$$

In this case, since the b_i are independent, each difference wx_i-zy_i must itself be 0, that is $wx_i=zy_i$, or

$$\frac{x_i}{z}=\frac{y_i}{w}.$$

Given an independent set $\{\ell_1,\dots,\ell_m\}$ of lengths and an independent set $\{\alpha_1,\dots,\alpha_n,\pi\}$ of angles, we define the product

$$\frac{1}{z}\left(\ell_1x_1+\cdots+\ell_mx_m\right)\otimes\frac{1}{w}\left(\alpha_1y_1+\cdots+\alpha_ny_n+\pi t\right)$$

as the sum of all of the
$$\ell_i \otimes \alpha_j \frac{x_i y_j}{zw}.$$
This definition satisfies our three rules of combining wedges, *and* it does not impose any integral dependency on the wedges $\ell_i \otimes \alpha_j$. Therefore these wedges are independent. In particular, none of them is nil.

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