Pappus's Hexagon Theorem

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Draft of August 17, 2026

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We prove the theorem named for Pappus of Alexandria about the intersection points or the parallelism of the opposite sides of a hexagon whose vertices lie alternately on two straight lines. Our proof uses only Book I of Euclid's *Elements,* and even only the propositions whose enunciations do not involve circles in any way. In particular, we develop a theory of proportion without using the Archimedean postulate that is implicit in Book V. This is roughly what Hilbert and Artin do, in interpreting a field in an affine plane. We try to give such a proof as might have been given in ancient times. Pappus proves several cases of Pappus's Theorem as well, as lemmas that would be needed for the remaining cases. Whether he had a synoptic view of what he was doing, we do not know for sure. Some modern researchers have not taken such a view of Pappus's results. As an example, Euclid must have had a synoptic view of his work in Book X of the *Elements,* even though he does not explicitly give such a view to us.

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1 Introduction

A hexagon has six vertices and thus three pairs of opposite sides. In a Euclidean plane, the members of each pair of opposite sides are either parallel or, when extended sufficiently, intersecting. There cannot be exactly two pairs of parallel opposite sides, if the vertices lie on a conic section; for, in this case, the following three implications hold, as shown by the hexagons $ABCDEF$ in the respective diagrams.

1. If each of two adjacent sides, such as AB and

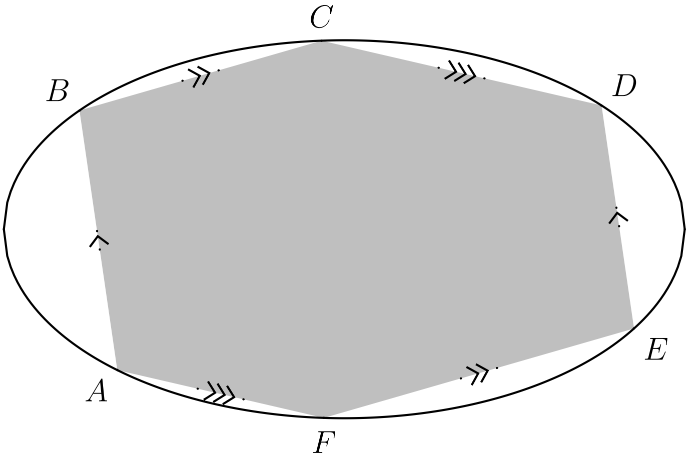

Figure 1: Hexagon in ellipse, parallel case

BC in Figure 1, is parallel to its opposite, then the remaining two sides, like CD and AF, are parallel to one another.

2. Of two adjacent sides, such as AF and AB in Figure 2, if one, like AF, is parallel to its opposite, but the other intersects its opposite at some point, like G, then somewhere on the straight line that passes through that point and is parallel to the former of the two adjacent sides, the remaining two sides intersect, as at

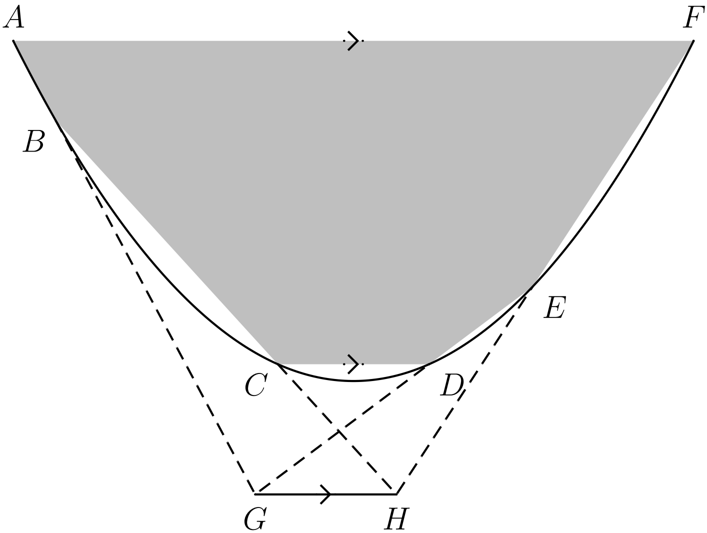

Figure 2: Hexagon in parabola: mixed case

H.

3. If each side intersects its opposite, then the three intersection points, like G, H, and K in Figure 3, lie on a straight line.

These three implications together (for a hexagon whose vertices lie on a conic section) are what we shall call the **Hexagon Theorem.** Individually, the implications will be the **parallel, mixed,** and **intersecting** cases, respectively.

The hexagon of the theorem may be non-convex, as we have seen in Figure 3. As in Figure 4, the hexagon may be non-simple, in the sense of having sides that intersect between their endpoints.

In another way, the Hexagon Theorem has the following two special or limiting cases:

Pappus's (Hexagon) Theorem, where the conic section is degenerate in the sense of being a pair of straight lines, which may be parallel or intersecting [31, 32, 33].

Pascal's Theorem, where the conic section is a circle [8, 34, 35, 48, 53].

From these two theorems, the whole Hexagon Theorem follows, at least if we allow ourselves to pass to a third dimension; for, briefly, an arbitrary nondegenerate conic section is the projection of a circle from a point not in its plane.

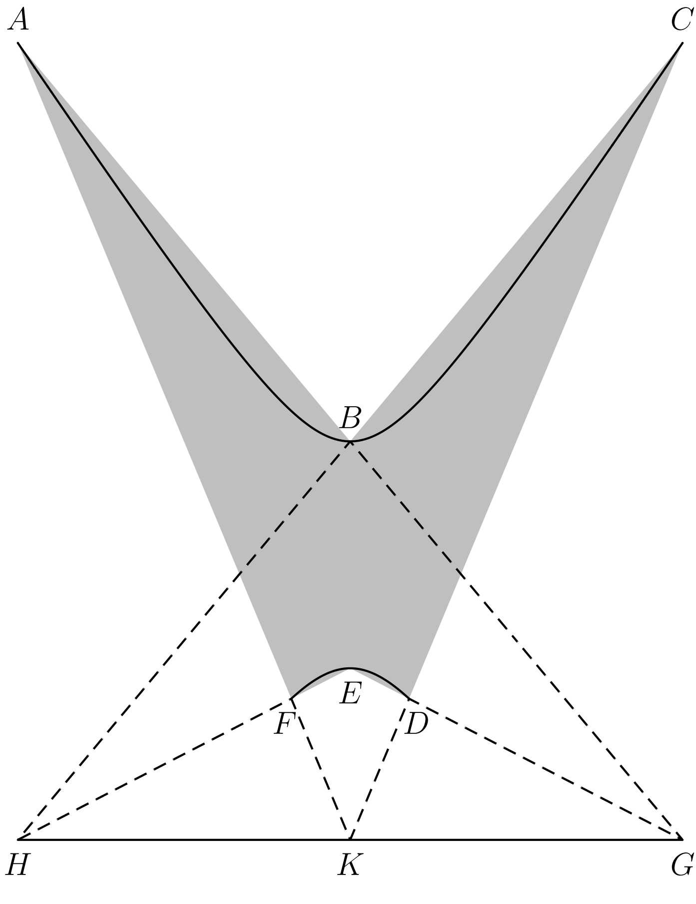

Figure 3: Hexagon in hyperbola, intersecting case

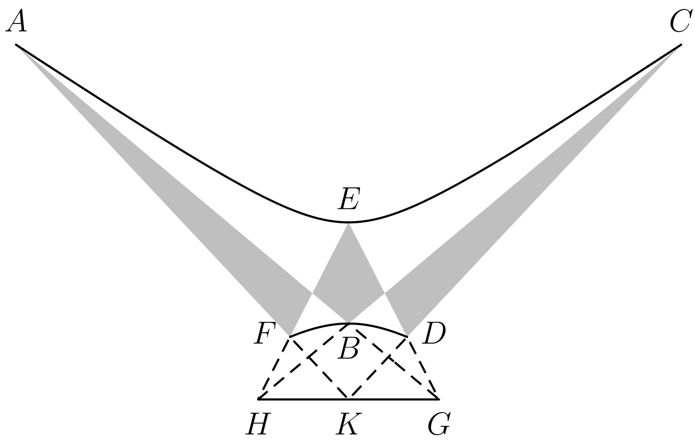

Figure 4: Non-simple hexagon in hyperbola

We are going to be interested in Pappus's Theorem, in a plane. It is a nontrivial result that we can and shall prove, using only the tools of Book I of the *Elements* of Euclid [18, 19, 20, 21]. Giving such a proof would seem to be more than Pappus himself did, and this is one reason to call his theorem nontrivial. We shall need only the *affine* tools from Euclid's Book I – roughly, the propositions not requiring circles for their enunciations.

In an affine plane, Pappus's Theorem has the parallel, intersecting, and mixed cases described for the Hexagon Theorem in general. The other two cases reduce to the intersecting case in a *projective* plane, obtained from an affine plane by addition of a "point at infinity" for each class of parallel straight lines, and a "straight line at infinity" comprising those new points. However, in a projective plane, Pappus's Theorem does not really follow from anything simpler. Indeed, for Coxeter [9, ch. 14, p. 231], Pappus's Theorem is among the *axioms* for a projective plane. This is another reason to call the theorem nontrivial for an affine plane.

Both the intersecting and the parallel cases of the theorem named for Pappus are found in Book VII of his *Synagôgê* (Συναγωγή) or *Collection.* They are among the lemmas written to aid the reader of the three books of the *Porisms* of Euclid. Those books must have lasted the six centuries between Euclid and Pappus, but unfortunately have since been lost.

The proof of the parallel case of Pappus's Theorem relies on such a theory of *areas* as is found in Book I of the *Elements.* The proof of the intersecting case relies on a theory of *proportion.* Presumably Pappus's theory is Euclid's, as developed in Book V of the *Elements.* That theory is "analytic," in the sense of using the "Archimedean" postulate.

We shall develop a theory of proportion that is purely geometric. This is roughly what Hilbert [26, 27] and Artin [3, 4] do. However, their theories are still somewhat "algebraic," and more so than need be.

We aim to follow more closely the ancient style. One reason to do this is to contemplate better the question of whether our theory *could* have been developed in ancient times.

As before, the hexagon that we consider will always be $ABCDEF$, except when we look at a new hexagon (in Figure 14 and 15) from an old diagram, or we look at one of Pappus's own diagrams (in Figure 17). When they exist, intersections of straight lines through pairs of vertices of $ABCDEF$ will always be labelled as in Figure 5, which shows what we may call the **fully intersecting** case. All of the possibil-

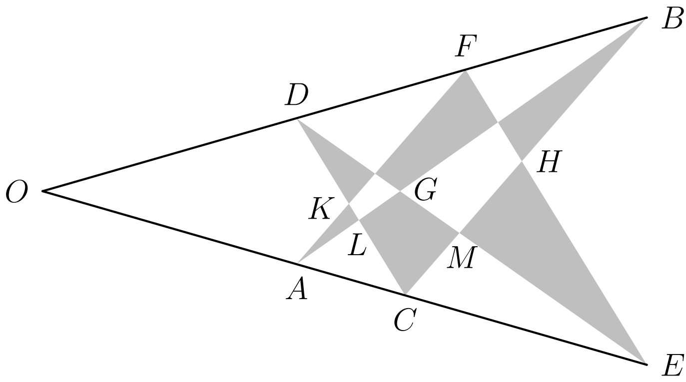

Figure 5: Pappus's Theorem, fully intersecting case

ities that need consideration are as in Table 1. The need for many diagrams need not be a burden, if we consider the diagrams aesthetically, as suggested in [42].

We may designate the fully intersecting case simply as $(1, 2)$, because O exists and both L and M exist. A case of no O would be $(0, -)$; no M, $(-, 1)$; no L and M, $(-, 0)$. Strictly speaking then, of the intersecting case, Pappus proves only the cases $(1, 2)$ and $(0, 2)$ explicitly. The other intersecting cases, along with the mixed cases, follow by means of lemmas that he proves.

Table 1: Configurations for Pappus's Theorem

O	G	H	K	L	M	figure	case
1	1	1	1	1	1	5, 28	intersecting $(1, 2)$
1	0	0	0	–	–	9, 12	parallel
0	0	0	0	–	–	10, 11	parallel
0	1	1	1	1	1	13	intersecting $(0, 2)$
1	1	1	1	1	0	30, 31	intersecting $(1, 1)$
0	1	1	1	1	0	32	intersecting $(0, 1)$
1	1	1	1	0	0	35	intersecting $(1, 0)$
0	1	1	1	0	0	36	intersecting $(0, 0)$
1	1	1	0	1	1	37	mixed $(1, 2)$
0	1	1	0	1	1	38	mixed $(0, 2)$
1	1	1	0	1	0	39	mixed $(1, 1)$
0	1	1	0	1	0	40	mixed $(0, 1)$

From third grade, I can remember being asked to corroborate one result found in Book I of the *Elements.* It is the thirty-second of the forty-eight propositions of the book. Each of us students was invited to

1) cut a triangle out of paper,
2) tear off the angles,
3) bring them together on one side of a straight line, and
4) see that they were together equal to a straight angle.

A child could likewise be asked to corroborate Pappus's Theorem, with a ruler (also Pascal's Theorem, with ruler and compass). The activity may suggest that there are laws to the universe. The existence of the law called Pappus's Theorem might confirmed in school at a convenient time, along the lines to be reviewed here.

The remaining sections of this paper are as follows.

§ 2. By Book I of the *Elements,* triangles on the same base are equal, just in case this base is parallel to the straight line through the apices. The triangles themselves are *equal*; the areas abstracted from them are the *same.*

§ 3. To define when two lengths have the same ratio as two other lengths, instead of relying on the

"Archimedean" postulate as Euclid does, we are going to use the *parallel* case of *Desargues's Theorem.*

§ 4. In the mathematical revolution wrought by Descartes, we may have forgotten things worth remembering. Descartes himself took inspiration from Pappus. He relied on a theorem found in Euclid, named now for Thales. This can serve as a *definition* of when ratios of lengths are the same. For this, we need parallel Desargues, to ensure that the "sameness" in question is indeed transitive. Meanwhile, in a *projective* plane, which *could* have been humanity's original object of mathematical study, Pappus's Theorem reduces to the intersecting case, but no longer follows from "obvious" assumptions.

§ 5. In a Euclidean plane, even an affine plane, we prove the parallel case of Pappus's Theorem using areas.

§ 6. That Pappus himself proved the parallel case has been overlooked in modern times. If *he* saw it as a case of one big theorem, he might well not say so, just as Euclid states no grand unifying theorem in Book x of the *Elements.*

§ 7. In a projective plane, the intersecting case of Pappus's Theorem is equivalent to the Fundamental Theorem of Projective Geometry, which is that a composite of central projections is determined by what it does to three points.

§ 8. In an affine plane, from parallel Pappus is derived parallel Desargues, by the method of Hessenberg. From parallel Desargues is derived the theorem that a composite of parallel projections is determined by what it does to two points.

§ 9. Now we can define ratios and when they are the same.

§ 10. Now we have Thales's Theorem, whence (1) Menelaus's Theorem, (2) of Pappus's Theorem in the *fully* intersecting case, and (3) a result of Pappus about *complete quadrangles.*

§ 11. We obtain, as limiting cases of the fully intersecting case, the other cases of the intersecting case of Pappus's Theorem, along with the mixed case.

§ 12. We have used some modern symbolism, instead of writing out everything in words, but Pappus too follows an abbreviated style.

2 Equality

Book 1 of Euclid's *Elements* is about *equality.* The first proposition is to construct a triangle that has three equal sides. The next two propositions let us, from a segment of a straight line, cut a part that is equal to a shorter segment of another straight line. In these propositions, the equality of different segments is traced ultimately to the equality of all segments drawn to the circumference of a circle from its center.

By Proposition 27 and 29, the alternate angles in which a transversal cuts two straight lines are equal, if and only if the two straight lines themselves are parallel.

A right angle is one of two equal angles formed on one side of a straight line when another meets it. The last two propositions of the book are the Pythagorean Theorem and its converse: an angle of a triangle is right, just in case the square on the opposite side is equal to the squares on the adjacent sides.

Perhaps one wants the latter condition to be that the *area* of the one square is equal to the sum of the areas of the other two squares. However, for Euclid, the one square itself is equal (hypothetically or potentially) to the other two squares. Equality is definitely not identity. We might however say that there *is* something, namely *area,* that is identical between the one square and the other two squares.

The fourth proposition of Book 1 is today's "Side Angle Side." For Euclid, this is that in two triangles, a certain set of three equalities yield *four* more. If, in respective triangles, two sides and the included angle are equal to two sides and the included angle, then the remaining side and two angles are equal to the remaining side and two angles, *and* the triangles themselves are equal.

That last equality does not imply the others. This becomes clear in Proposition 35: parallelograms on the *same* base in the *same* parallels are equal to one another, even though they are usually incongruent. Thus equality is not even congruence, much less sameness.

It is elsewhere said that Euclid's equality *is* congruence [39, pp. 104–6, 137]. This is so for angles and for segments of straight lines. For triangles, congruence is only *sufficient* for equality, as in Proposition 4.

This is not exactly the understanding of Heath as translator and commentator: he says in his note on Proposition 35 [19, i.327],

> It is important to observe that we are in this proposition introduced for the first time to a new conception of equality between *figures*. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I. 4).

More precisely, equality is *inferred* from congruence in Proposition 4; in Proposition 35, the same equality

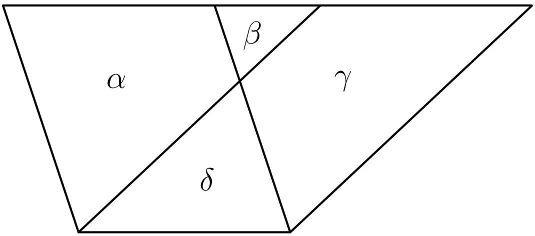

Figure 6: Proposition 35 of Book I of Euclid's *Elements*

is called on, but then there is some arithmetic or algebra, so to speak. In Figure 6, the three equalities that constitute the hypothesis of Proposition 4 are satisfied by the two overlapping triangles whose parts are labelled respectively as α and β and as β and γ. Thus

$$\alpha + \beta = \beta + \gamma. \tag{1}$$

Then $\alpha = \gamma$, by the "common notion" that equals subtracted from equals leave equal remainders. By the corresponding common notion for addition,

$$\alpha + \delta = \delta + \gamma. \tag{2}$$

It may be misleading to write out our equalities as in (1) and (2). Presumably we have for example

$$\alpha + \beta = \beta + \alpha;$$

however, what would each member of the equation mean? One polygon can be added to another, not just from either side, but from any direction, and with any orientation, without affecting the area of the result.

In Book 1 of the *Elements,* In the passage from Proposition 4 to 35, the *meaning* of equality has not changed, even though Heath continues his note as follows.

> Now, without any explicit reference to any change in the meaning of the term, figures are inferred to be *equal* which are equal in *area* or in *content* but need not be of the same *form.* No *definition* of equality is anywhere given by Euclid . . .

Again, I would say that while congruent triangles are themselves *equal,* as are the polygons that can be derived from them as in Proposition 35, the *areas* of those equal polygons are not equal, but the *same.*

The preceding Proposition 34 is that a parallelogram

1) has equal opposite sides and angles and
2) is bisected by a *diagonal* (διάμετρος “diameter”).

The latter part and Proposition 35 yield 37: *triangles* on the same base and in the same parallels are equal.

We can use the latter part of Proposition 34 alone to establish (1), thus Proposition 35 and 37. This is

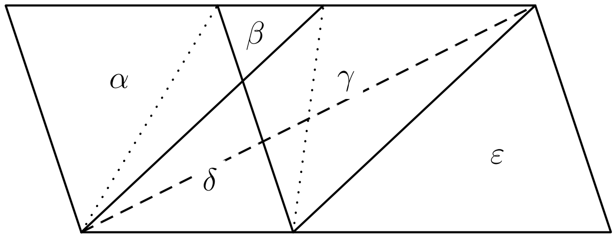

Figure 7: Bisection of parallelograms

shown by Hähl and Weber [28, § 4], albeit with no mention of Euclid. In Figure 7, the dashed straight line bisects each of *two* parallelograms, yielding the equality of the differences of the equal halves:

$$\alpha + \beta = \varepsilon.$$

Likewise, because they are the two halves of a third parallelogram,

$$\varepsilon = \beta + \gamma.$$

This gives us (1), from which, as before, we obtain (2), which is the equality of the parallelograms whose diagonals are now the dotted straight lines. Thus we have the equality of halves of *those* parallelograms – for example, of the halves that are the triangles that share the base of δ. (We are assuming, as Hähl and

Weber do explicitly, that the common area of the parallelograms does not have order two in the *group* of areas).

By Propositions 44 and 45, every polygon is equal to a rectangle on a common base. Actually, Euclid makes this more general, for no obvious reason: the rectangle can instead be a parallelogram in any angle. The common base corresponds to

- the unit length of Descartes that we shall mention in § 4;
- the fixed length with respect to which, in Book x of the *Elements,*
  - the **rational** (ῥητόν) area is *commensurable* (σύμμετρον) with the square on that length;
  - the **irrational** (ἄλογον), not;
  - the **medial** (μέσον) area is irrational, but is a *mean proportional* (μέσον ἀνάλογον) of two rational areas.

We shall look at what Euclid does with these definitions in § 6. Meanwhile, for no reason other than intrinsic interest, I want to record some observations involving Euclid's terminology. The Greek adjectives reflect the neuter gender of χωρίον "area," a word that (like the feminine γραμμή "line") is rarely used explicitly. The English "mean proportional" seems *not* to originate as a word-for-word translation of μέσον ἀνάλογον. If it did, its plural ought to be "means proportional," just as the plural of "Whopper Junior" should be "Whoppers Junior," according to a satirical article in *The Onion* [46]. The Greek term in question is μέση ἀνάλογον for one straight line, μέσαι ἀνάλογον for several straight lines; ἀνάλογον "proportional(ly)" is an adverb originating in a prepositional phrase. In "mean proportional," the second word is usually construed as a noun, though it can be an adjective in isolation. Indeed, in the *Oxford English Dictionary* [29], the earliest quoted *mathematical* use of "proportional" as an adjective is from Billingsley's 1570 translation of Euclid's *Elements* [17, folio 130 verso]:

> Magnitudes which are in one and the selfsame proportion, are called <u>Proportionall</u>.

In place of Billingsley's "proportion," we use "ratio," as in Heath's translation. The *OED*'s earliest quoted use of "proportional" as a noun is from John Dee's "Mathematicall Praeface" to Billingsley's translation [12, folio c.iij verso]:

> Betwene two lines giuen, finde two <u>middle proportionals</u>, in Continuall proportion: by the hollow Parallelipipedon, and the hollow Pyramis, or Cone.

Here "middle" could presumably be "mean."

3 Proportion

Two areas, or two lengths, or two volumes, may have the *same ratio* as two *other* areas, lengths, or volumes. However, Euclid never describes two ratios as being *equal* [39, pp. 97, 115, 136–8]. A proportion is thus not an *equality* of ratios, but an identity.

Euclid's usage is important here, particularly when it comes to interpreting the definition, in Book VII of the *Elements,* of when two *numbers* are in the ratio of two other numbers. A rule such as

$$a : b :: c : d \iff ad = bc \tag{3}$$

is no good for a definition, because the condition on the right does not *obviously* distinguish anything about a and b that is the same as it is for c and d. In particular, by (3) alone, the relation of sameness of ratio is not obviously transitive. Euclid's definition is effectively this: for the left side of (3) to hold, what we call the Euclidean Algorithm (which is set out in the first three propositions of Book VII) must (and need only) have the *same steps,* whether applied to a and b or to c and d [41, § 4]. This condition is that, however many times b goes into or *measures* a, so many times d goes into c, and so on with the remainders. This yields that each of a and b is respectively the same multiple of the greatest common measure of the two that each of c and d is of *their* greatest common measure.

I said that we were going to prove the Hexagon Theorem using only Book I of the *Elements,* and that the book was about equality. The Hexagon Theorem is not obviously about equality. However, the proof of the parallel case will need from Book I only Proposition 37, which we saw in § 2, and its converse, Proposition 39.

In the proof of the intersecting case of Pappus's Theorem, ratios will come up. The ratio of two magnitudes is a measure of how far they are from being equal. To have a ratio, unequal magnitudes should still be *comparable,* in the sense of being lengths, areas, or volumes *alike.*

In the "Eudoxan" theory, established by Euclid in Book V of the *Elements,* but attributed to Eudoxus in a scholium [49, pp. 408f.], magnitudes that have a ratio must be comparable in a stronger sense: some multiple of the less must exceed the greater, as is the case for numbers. If two lengths, areas, or volumes are *always* to have a ratio, then the Eudoxan theory requires the postulate that is named for Archimedes, even though he was working after Euclid (not to mention Eudoxus).

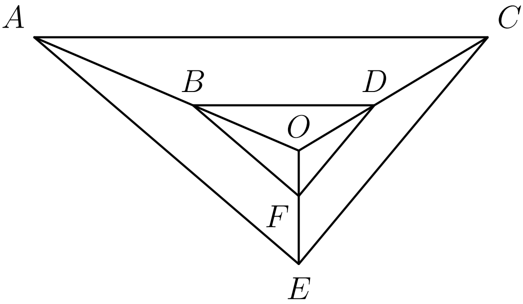

Figure 8: Desargues's Theorem, parallel case

To do geometry, we do not actually need the Archimedean postulate. We know this from Hilbert's *Foundations of Geometry* [26, 27]; but again, we are going to work more explicitly in terms of Book I of the *Elements.* Briefly, a ratio will be a certain equivalence class of ordered triples of points of straight lines. The key requirement is that if (O, A, B) and (O, C, D) are equivalent, but in different directions, then AC and BD must be parallel. The transitivity of the equivalence relation will follow from the parallel case of **Desargues's Theorem:** in Figure 8, if AC and BD are parallel, as are AE and BF, then so are CE and DF. We can prove this theorem from the parallel case of Pappus's Theorem, thanks to Hessenberg [25].

In a sense, we shall be following Artin [3, 4], but in more elementary terms. For him, the role of our ratios is played by the "trace-preserving homomorphisms" of the "group of translations." (A translation takes the points of any straight line to the points of a parallel straight line, fixing either all points or none, and in the latter case, the lines fixed setwise are the traces.) Since Artin prefers not to require parallel Pappus to be true, he introduces parallel Desargues as an axiom. *Our* axioms are whatever is enough to give us Propositions 37 and 39 of Book 1 of the *Elements*: again, that triangles on the same base are equal if and only if the straight line through their apices is parallel to the base.

4 Some history

Reading Euclid and Pappus, we may see better what has been done *for* and *to* mathematics by Descartes [13, 14], through the introduction of methods that we describe as symbolic, algebraic, and analytic.

More than three centuries before Descartes wrote the *Geometry,* Dante wrote the *Divine Comedy,* in which the universe is what we now call a 3-sphere. We obtain this, in thought, by gluing together, at their surfaces, two 3-balls, namely

- the earth, with a fallen angel at the center, and
- the heavens, with the Deity at the center.

Such a conception, without the theology, had to be discovered anew by modern physics, because meanwhile, space had come to be conceived of, both as being different from whatever was in it, and as being Euclidean. This is by the account of William Egginton [16], as I understand it. Egginton's reasonable conclusion is that we do not always know better then we used to:

> the general assumption is that knowledge is cumulative and that the seventeenth century had more of it than the Middle Ages. I have suggested in this essay that this is not always the case . . . a culture's framework for viewing the world may be forgotten. With splendid self-assurance the western empirical tradition assumes that such forgetting has only a utilitarian character: one only loses what one no longer needs . . . We need to recognize that sometimes the very gains that open new possibilities of thought can, at the same time, foreclose other possibilities, ways of thinking, and forms of imagination that future ages will need to rediscover.

I have been rediscovering Pappus since noticing that the Wikipedia article "Pappus's Hexagon Theorem"

[55] said nothing about his own proof of the theorem. (It does now, because I added the section called "Origins" – as I point out, because you can do the like, when you find an inadequate article.)

All cases of Pappus's Theorem collapse into the fully intersecting case in a projective plane. In *Introduction to Geometry* (1969), when Coxeter describes what is interesting about a projective plane, he seems not to recognize what Euclid must have done [9, ch. 14, p. 229]:

> we might expect a geometry having no circles, no distances, no angles, no intermediacy, and no parallelism, to be somewhat meagre. But, in fact, a very beautiful and intricate collection of propositions emerges: propositions of which Euclid never dreamed, because his interest in measurement led him in a different direction. A few of these nonmetrical propositions were discovered by Pappus of Alexandria in the fourth century A.D. Others are associated with the names of two Frenchmen: the architect Girard Desargues (1591–1661) and the philosopher Blaise Pascal (1623–1662) . . .

Coxeter thus seems unaware that Pappus's proof of Pappus's Theorem comes in the part of the *Collection* that is devoted, as we said, to the three books of *Porisms* of Euclid.

For the projective plane, as we noted in § 1, Coxeter uses Pappus's Theorem as an *axiom.* Perhaps it is not a very intuitive axiom, unless one has seen it developed in the Euclidean plane by means of measurement in the sense already suggested.

Euclid infamously defines a *straight* line as a line (which otherwise may be curved) that lies evenly with the points of itself. Lucio Russo contends that the definition is really a quotation from the *Definitions* attributed to Heron, and that the quotation was appended only later to Euclid's manuscript, but truncated so as to be meaningless [45, pp. 320–7]. If we do have the definition as Euclid intended, then it could mean that a line, or a segment of one, is straight, precisely when it can be *sighted* along. This means all other points fall behind one of them, as other people may fall behind one of them in a queue at a bus stop.

Such an understanding of straight lines would indicate a *tactile* geometry, concerning things that we can *manipulate.* That is what geometry is originally anyway, since the first meaning of γεωμετρία is surveying. This is seen in Herodotus, who tells us that the Greeks learned "geometry" or surveying from the Egyptians, who needed to remeasure their fields each year, after the flooding of the Nile [24, 2.109].

Nonetheless, we can also judge whether a line is straight, just by looking at it from the side. When the sun is behind clouds, we may still see its rays, emanating as straight lines. The same is true when we look at our shadow in the sea: straight rays converge on darkness. If we had eyes on either side of our heads like a chicken, then we might have both visions at once, seeing many straight lines, all passing between the same two antipodal points. If we conceived of every pair of such points as one point, then we would turn the celestial sphere into a projective plane.

Thus, we can imagine that our first object of "geometrical" study could have been the projective plane. This might be sufficient reason for studying Pappus's Theorem as it is in the projective plane. However, this is not what we are going to do here.

In *The Foundations of Geometry* (fourth edition, 1963), Gilbert de B. Robinson has a neat axiomatic exposition of projective geometry [44, pp. 9–41], where Pappus's Theorem becomes Axiom IX, ultimately interchangeable with the *Fundamental Theorem of Projective Geometry.* As Robinson notes,

> Pappus of Alexandria (circ. 340 A.D.) gave a proof of the theorem which we have associated with his name, under the assumptions of Euclidean geometry (cf. chapter V); but *no proof is possible on the basis of axioms* I–VIII.

Again, what is for us Pappus's Theorem is effectively three theorems, all set in the Euclidean plane. Pappus leaves out the mixed case. In proving the intersecting and mixed cases, we shall use even fewer assumptions than Pappus does.

As already noted in § 1, we shall not use Book v of the *Elements.* This book is "analytic," at least in the sense of belonging to what today is called real analysis. As we noted in § 3, the book relies on the assumption that some multiple of the less of two lengths, areas, or volumes always exceeds the greater. Archimedes states the assumption as one of his *postulates* (λαμβανομένα "things taken") in the first book *On the Sphere and the Cylinder* [1, pp. 8, 10], [2, pp. 36, 40]. Euclid's explicit *postulates* (αἰτήματα "demands") are only the five in Book i [18, p. 8]. Euclid makes the "Archimedean assumption" in Book v, but not explicitly as a postulate. Because of it, each of Euclid's ratios can be identified with a positive element of the *field of real numbers.*

Euclid did *not* ensure conversely that each positive real number would be a ratio of magnitudes. He *could* have ensured this, by demanding that for every division of a straight line into two pieces, there should be a point effecting the division. In my memory of high-school geometry, we students were given such a postulate; however, I have not found it in the text-book that we used [54]. In any case, Dedekind

1) formulated such a postulate,
2) proved it as a theorem about the particular straight line whose points are *cuts* of rational numbers, and
3) clarified that his work was indeed about numbers, not quantities – it was not geometric, but arithmetic [11, pp. 11, 20f., 36–40].

As suggested in § 1, geometry without the Archimedean assumption is developed by Hilbert [26, 27] and then Artin [3, 4]. They show how to interpret a *field* in a Euclidean plane and even an **affine plane:** a Euclidean plane without circles, thus without equality of segments of straight lines that are not parallel. For Hilbert, as for Descartes before him, the elements of the field are lengths of segments of a straight line; for Artin, as noted in § 3, they are trace-preserving homomorphisms from the group of translations into itself. Whether the field is commutative or not depends on the parallel case of Pappus's Theorem.

We shall prove the parallel case as Pappus himself does, but on the basis of the theory of areas in Book I of the *Elements.*

Perhaps because the letters in Descartes's polynomial equations all stand for lengths, today we may overlook the value of areas in doing "pure, synthetic," or "axiomatic" geometry (rather than differential geometry). In an example *not* investigated further here, Apollonius gives two characterizations of conic sections: by equations

1) of square with rectangle and
2) of triangle with parallelogram or trapezoid.

With the latter equation, he neatly establishes what we would call a change of coordinates [40, 41].

Not only does Descartes use letters for lengths, but he assumes that a specific *unit* length has been chosen [14, p. 2]. This allows him to define the product of two lengths as itself a length (this is discussed and made use of in [37]). The definition uses what is called **Thales's Theorem** in some countries [36] and by us, namely Proposition 2 of Book VI of Euclid's *Elements*: a straight line cutting two sides of a triangle cuts them in the same ratio if and only if the cutting straight line is parallel to the third side of the triangle. The result follows from Proposition 1 of the same book, that two rectangles of the same height are in the ratio of their bases, and so are two triangles. The proof of *this* relies on Book V. We cannot take Proposition 1 or 2 of Book VI as a *definition* of sameness of ratio unless we know that the relation is going to be transitive. We shall know this, as we said in § 3, by proving the parallel case of Desargues's Theorem.

Descartes is inspired by the discussion of the *locus problem* at the beginning of Book VII of the *Collection.* Given $2n$ straight lines in a plane, we may ask for the locus of points, the product of whose distances to the first n of the straight lines bears a given ratio to the product of the distances the last n straight lines. If the product of two lengths is an *area*; and of three lengths, a *volume*; then it is not clear what the product of four or more lengths should be. Descartes gets around this problem by defining the product of two lengths as a length. However, he also quotes Pappus's earlier solution to the same difficulty: the ratio of the two products of n lengths each can be understood as the product, or *composite,* of the n ratios of the respective factors. Since ratios are dimensionless, any number of them can be composed.

5 Parallel Pappus

In the manner suggested in § 4, we may consider the parallel case of Pappus's Theorem as two:

trapezoidal, as in Figure 9, and

parallelogrammatic, as in Figure 10.

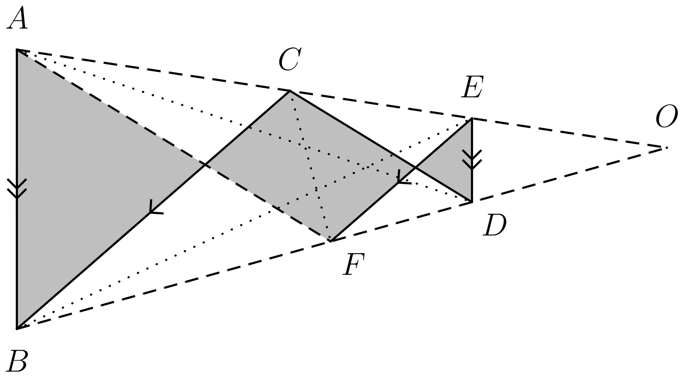

Figure 9: Pappus's Theorem, trapezoidal case

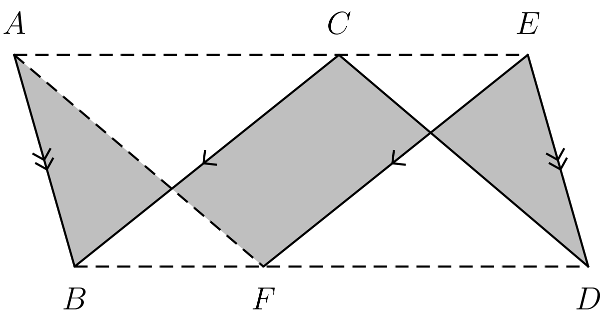

Figure 10: Pappus's Theorem, parallelogrammatic case

In either diagram, the quadrilateral $ABDE$ is of the indicated kind, while the hexagon $ABCDEF$ has vertices lying alternately on the straight lines AE and BD, and

$$BC \parallel FE, \qquad BA \parallel DE. \tag{4}$$

If AE and BD do intersect, it is at O. In either case, the desired conclusion is

$$AF \parallel CD, \tag{5}$$

Pappus's Lemma VIII is the trapezoidal case. He leaves out the parallelogrammatic case, but it will follow from the theorem that a quadrilateral with two parallel sides is a parallelogram if and only if those sides are also equal. This theorem is Proposition 33 of Book I of the *Elements*, along with the part of Proposition 34 that we did not consider further in § 2. In Figure 10, the quadrilaterals $ABDE$ and $BFEC$ being parallelograms,

$$AE = BD, \tag{6}$$
$$CE = BF. \tag{7}$$

Therefore, by subtraction,

$$AC = FD, \tag{8}$$

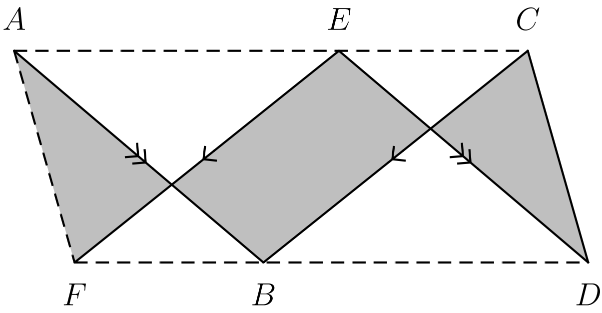

Figure 11: Pappus's Theorem, parallelogrammatic case

so $AFCD$ is a parallelogram. If our diagram is as in Figure 11, then we may write (7) as

$$EC = FB$$

and *add* the members to those of (6), so as to obtain (8). In either case, we are only applying the rule whereby, regardless of order,

$$AE - CE = AC = AE + EC.$$

Perhaps such a rule is thoroughly modern. On the other hand, Pappus would seem to understand that a different order of points on a straight line does not require a different proof of the associated result. For example, for the first of the lemmas for Euclid's *Porisms,* Pappus gives five diagrams: at least, five are found in the manuscripts [31, p. 867], though they are not described in words [33, p. 457].

To establish the trapezoidal case, in Figure 9, we draw the auxiliary straight lines AD, BE, and CF. We apply *Elements* 1.37, which we proved in § 2, and 39, its converse, as we said in § 3. By the parallelisms in (4), we have respectively

$$EFC = EFB, \qquad EDB = EDA. \tag{9}$$

To the members of these equations, we add respectively OFE and ODE, obtaining

$$OFC = OBE, \qquad OBE = ODA$$

and therefore $OFC = ODA$. Subtracting ODC yields

$$DFC = DAC \tag{10}$$

and then (5). Again, the order of points does not matter; it could be as in Figure 12. In any case,

$$EFC + OFE = OFC = EFC - EFO,$$

and so forth.

We can even make one proof work for both the parallelogrammatic and trapezoidal cases. We have the

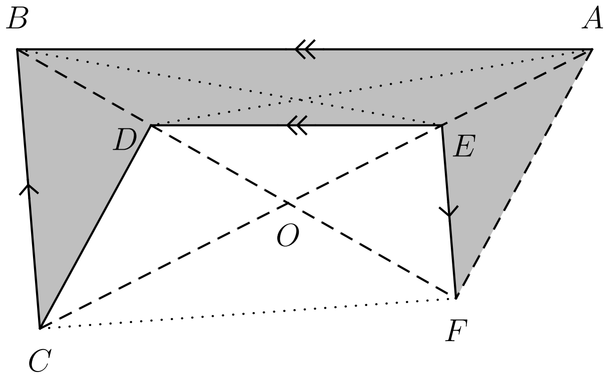

Figure 12: Pappus's Theorem, trapezoidal case

equations of (9) in Figure 10 and 11, as well as in 9 and 12. Adding DFE to the members of the former equation of (9) yields

$$DFCE = DBE,$$

although the quadrilateral $DFCE$ may not be simple. From this and the latter equation in (9), we obtain

$$DFCE = DAE.$$

Subtracting DCE leaves us with (10).

6 The parallel case in history

Pappus's actual Lemma VIII uses the configuration of Figure 17 below, where

$$\mathsf{\Delta E} \parallel \mathsf{B\Gamma}, \qquad \mathsf{EH} \parallel \mathsf{BZ}.$$

The Lemma as such has been curiously neglected. In his *Foundations of Geometry,* Hilbert does note in effect, as we have, that *Elements* I.37 and 39 yield the trapezoidal case of Pappus's Theorem [26, §21, p. 69]. In a later edition of the *Foundations,* Hilbert shows explicitly how to prove that trapezoidal case [27, §21, p. 69]. However, he may not actually have *read* Pappus, since he names the result in question for Pascal.

In his *History of Greek Mathematics,* 1921, when he reviews Pappus's "Lemmas to the 'Porisms' of Euclid" [23, pp. 419–24], Heath does not take note of Lemma VIII, though he names *four* other lemmas as involving Pappus's Theorem [23, p. 421]:

> The Lemmas 12, 13, 15, 17 (Props. 138, 139, 141, 143) are equivalent to the property of the hexagon inscribed in two straight lines, viz. that, if the vertices of a hexagon are situate, three and three, on two straight lines, the points of concourse of opposite sides are in a straight line; in Props. 138, 141 the straight lines are parallel, in Props. 139, 143 not parallel.

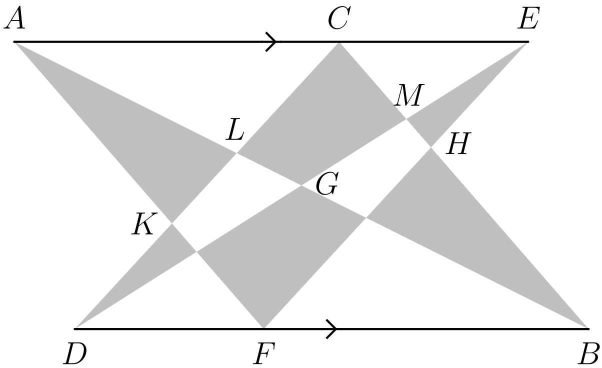

Figure 13: Pappus's Theorem, intersecting case $(0, 2)$

The numbering here of the lemmas and propositions of Pappus is as in Hultsch's Greek edition of the *Collection* [31]. Of this, Book vii – the one with the lemmas for the *Porisms* of Euclid – seems not to have had a full English translation before Jones's of 1986 [33].

We can apply

Lemma xii and xv to Figure 13, where $AE \parallel DB$;

Lemma xiii and xvii to Figure 5, where AE and BD intersect at O.

In either figure,

- AB and EF intersect at H,
- CD and FA intersect at K.

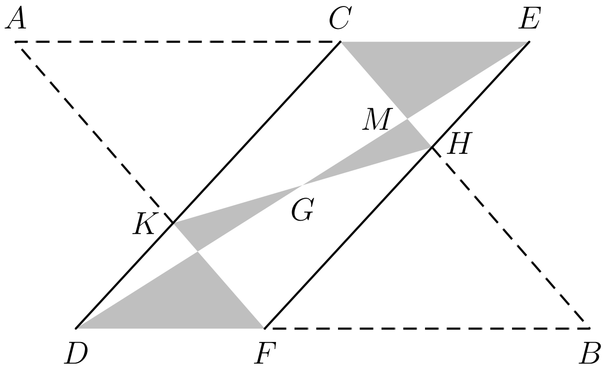

Figure 14: Lemma xv of Pappus

The conclusions are as follows.

Lemma xii and xiii. The intersection point G of AB and ED lies on HK.

Lemma xv and xvii. The intersection point G of HK and ED lies on AB.

This is not what Pappus does, but we can understand the latter conclusion as being the same as the former, applied not to the hexagon $ABCDEF$, but to $DECHKF$. This is the hexagon shaded in Figure 14 and 15, which have the same straight lines, and labels (except L) at intersections, as Figure 13 and 5 respectively. However, Pappus's proof of Lemma xii

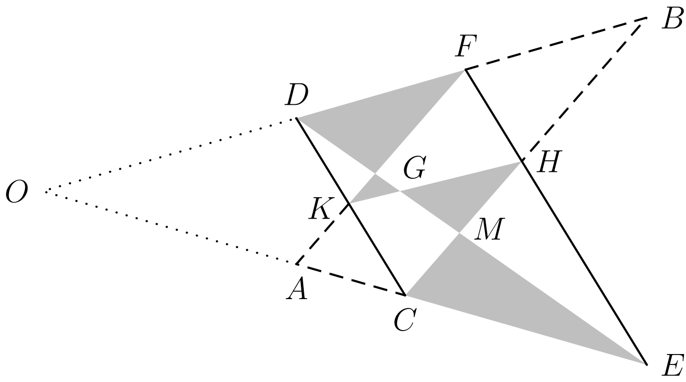

Figure 15: Lemma XVII of Pappus

and XIII needs the intersections labelled L and M in Figure 13 and 5, and the corresponding intersections may not exist in Figure 14 and 15. We shall take care of all cases later.

The words of Heath quoted above are an unacknowledged translation of a comment of Chasles in his treatise of 1860 on *Les trois livres de porismes d'Euclide, d'apres la notice et les lemmes de Pappus* [7, p. 77]. Heath has quoted from this work explicitly, in translation, at the beginning of his review of the lemmas of Pappus. Regarding the Hexagon Theorem, what Chasles says is,

> Les quatre Lemmes XII, XIII, XV et XVII (propositions

> 138, 139, 141, 143) peuvent être considérés comme exprimant la propriété de l'hexagone inscrit à deux droites, savoir que, quand les sommets d'un hexagone sont situés trois à trois sur deux droites, les points de concours des côtés opposés sont en ligne droite.

Thus the omission of Lemma VIII as a case of the Hexagon Theorem goes back to Chasles. The omission is seen also in Pambuccian and Schacht, "The Axiomatic Destiny of the Theorems of Pappus and Desargues," 2019 [30]:

> Pappus's Theorem can be found in Book VII of Pappus's Collection, where it is spread over Lemma XII, XIII, XV, and XVII in the part of Book VII containing lemmas to the first book of Euclid's Porisms (see [33, pp. 270–273] and [32]). It thus goes back to the first half of the fourth century A.D.

The article itself is concerned with

> the largely twentieth century history of the discovery of the significance of Pappus and Desargues for the axiomatics of geometry . . .

For this, there are 278 references, one of which is [28], which we used in § 2.

Of Pappus's thirty-eight lemmas on Euclid's *Porisms,* Chasles distinguishes the first twenty-one, along with Lemma XXXII and the last of the 38, as concerning rectilineal figures. Four of those twenty-three lemmas are said, as above, to express the Hexagon Theorem, but Lemma viii is classified among four miscellaneous lemmas that “se rapportent à certains systèmes de droites que nous définirons plus loin” [7, p. 74]. He says specifically later [7, p. 78],

> Le Lemme VIII (proposition 134) a un énoncé très-bref, qui en laisse difficilement apercevoir le sens; cependent on reconnait qu’il peut signifier que:
>
> *Quand deux angles ont leurs côtés parallèlles deux à deux, si par le sommet de chacun d’eux on mène une droite quelconque qui coupe les deux côtés de l’autre, les quatre points d’intersection sont deux à deux sur deux droites parallèles.*

Thus in Figure 17, if angles ΒΖΔ and ΕΗΓ are taken to have respectively parallel sides, then the points Ε and Γ where a straight line through the vertex of ΒΖΔ cuts the sides of ΕΗΓ are the endpoints of parallels whose other endpoints are respectively Δ and Β, where a straight line through the vertex of ΕΗΓ cuts the sides of ΒΖΔ. Chasles considers this to be a special case (*cas particulière* – strictly it is a limiting case) of a result in his *Traité de géométrie supérieure* of 1852 [6, ¶ 404, pp. 295f.]. At the back of the book, the diagram for this result is as in Figure 16. Here, the the ver-

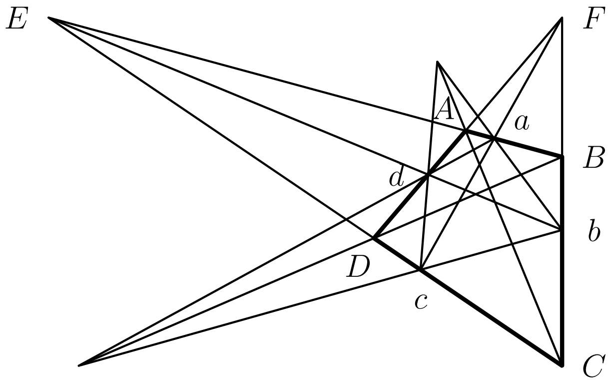

Figure 16: Chasles's Theorem

tices of the quadrilateral $ABCD$ are where the sides of the angle at E cut the sides of the angle at F. Random straight lines through the vertices of the angles cut the the sides of the angles in the vertices of the quadrilateral $abcd$. The conclusion is that the opposite sides of this quadrilateral intersect along the diagonals of $ABCD$. Chasles proves this independently, but it follows from Pappus's Theorem, applied to the hexagons $cEadFb$ and $cEabFd$, each of whose vertices lie alternately on the two random straight lines.

There are excerpts from Pappus's *Collection,* and Book VII in particular, in each volume of Thomas's anthology [49, 50]; however, from the lemmas for Euclid's *Porisms,* Thomas includes only Lemma IV, describing it as being [50, pp. 610–3]

> one of the ways of expressing the proposition enunciated by Desargues: *The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution* . . . A number of special cases are also proved by Pappus.

The ellipsis is for a reference to Cremona, who, in his second edition at least [10, pp. 106–8], has "quadrangle" where Thomas has "quadrilateral." Field and Gray review Pappus's Lemma IV (calling it "his fourth porism," and calling the figure "a complete quadrilateral") in *The Geometrical Work of Girard Desargues*; they too note the similarity with what they call Desargues's Involution Theorem [22, pp. 10–1, 54–5]. We shall come back to complete quadrangles at the end of this section.

Pappus's *Collection* has the 1933 French translation by Ver Eecke, whose extensive notes follow Chasles in connecting Lemma VIII to the equivalent result in Chasles's *Traité,* but not explicitly to the Hexagon Theorem [32, pp. 680–8].

Again, Pappus's own diagram for Lemma VIII is as in Figure 17; at least, it would be thus, before the construction of the auxiliary straight lines ΒΕ, ΓΔ, and ΖΗ. Pappus does not describe the construction of the

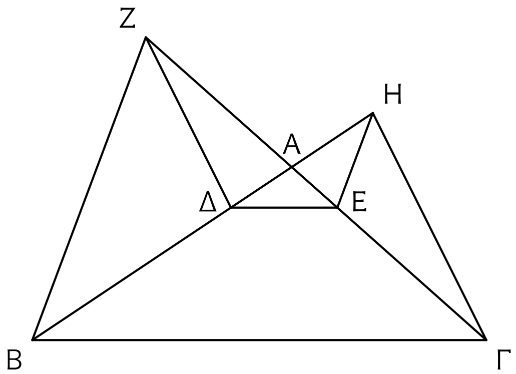

Figure 17: Pappus's diagram for Lemma VIII

other straight lines, beyond stating the parallelism of ΔE with BΓ and EH with BZ; otherwise, he says only,

Ἔστω βωμίσκος ὁ ΑΒΓΔΕΖΗ,
Let there be an "altar" ΑΒΓΔΕΖΗ.

Van Eecke gallicizes the Greek term as *bômisque,* explaining in a footnote that Heron of Alexandria used the term in the *Stereometry* for a hexahedron with two parallel dissimilar rectangular faces, the other four then being trapezoids.

With his own translation, Jones makes the connection that we have been looking for. He observes [33, p. 562] that Lemma VIII

> can be interpreted as a limiting case of Pappus's theorem, in which the hexagon (ΒΖΔΕΗΓ in [Figure 17]) has opposite sides parallel, so that their intersections lie on the straight line at infinity; it is an open question whether Euclid recognized the affinity between these propositions. [Footnote:] I owe these observations to Dr [Jan] Hogendijk.

Perhaps indeed the question of whether Euclid and Pappus recognized this affinity must be considered as open.

Euclid does not make explicit the affinities among propositions of the *Elements*, even though he must know them. For example, in Book x, Propositions 36–111 are all about what happens when we take two incommensurable areas, each rational or medial as defined in § 2, and we find the square equal to their sum or difference. If the two areas are X and Y, the former the greater, we can analyze them, uniquely, as $A+B$ and $2C$ respectively, where C is the mean proportional of A and B. If $A = d^2$ and $B = e^2$, then $X \pm Y = (d \pm e)^2$, and $d+e$ and $d-e$ belong in one of the six numbered cells of Table 2, where $A \equiv B$ means A and B are commensurable. In the first cell, a sum is called **binomial** (ἐκ δύο ὀνομάτων "of two names"); a difference, **apotome** ἀποτομή ("[straight line] cut off"). We shall not need the names of the

Table 2: Sums and differences

$d \pm e$	$A + B$ rational	$A + B$ medial	
	C medial	C rational	C medial
$A \equiv B$	1	2	3
$A \not\equiv B$	4	5	6

Table 3: Binomials and apotomes

$\zeta \pm \xi$	$\zeta \equiv \xi$	$\zeta \not\equiv \xi$	
		$\eta \equiv \xi$	$\eta \not\equiv \xi$
$\zeta \equiv \vartheta$	1	2	3
$\zeta \not\equiv \vartheta$	4	5	6

other lengths.

Fixing a length ξ on which the square is rational, we define ζ, η, and ϑ so that

$$X = \xi\zeta, \qquad Y = \xi\eta, \qquad \zeta^2 - \eta^2 = \vartheta^2.$$

Then $\zeta + \eta$ or $\zeta - \eta$ is an nth binomial or apotome, where n is as in Table 3. Now $d \pm e$ belongs in cell n of Table 2 if and only if $\zeta \pm \eta$ belongs in cell n of Table 2. Moreover, X and Y can be recovered from ζ

and η, and the latter two can be any incommensurable lengths on which the squares are rational.

That is what Euclid shows, but he doesn't say it that way. He just gives us 76 propositions, mostly falling into groups of six, corresponding to the six cells in one of the tables. He effectively gives us the appliance manual of William Thurston [51]:

> Mathematics in some sense has a common language . . . This language efficiently conveys some, but not all, modes of mathematical thinking . . . It's like a new toaster that comes with a 16-page manual. If you already understand toasters and if the toaster looks like previous toasters you've encountered, you might just plug it in and see if it works . . .

David Bessis remarks on this [5, p. 52],

> I couldn't find my toaster manual, but I did find the one for my vacuum cleaner. It's sixty-four pages long and never says what you use a vacuum cleaner for. In other words, if we were to stick strictly to the official literature, vacuum cleaners would remain inexplicable. Nobody would know what to do with them, except the few folks who'd have received the "gift of vacuuming" (*that is,* they'd have accidentally come up with a proper way to use them).
>
> The hidden sense of vacuum cleaners isn't found in the manual. It's a secret we pass on by word of mouth.

The secret of ancient mathematics has not been so passed on; or it may have been distorted, as in the telephone game.

It would seem fair to conclude that Pappus recognized an affinity, at least among the four lemmas that we looked at earlier, XII, XIII, XV, and XVII, even though straight lines that intersect in two of them are parallel in the others.

Chasles is concerned with making affinities explicit. Thus he distinguishes Lemma I, II, IV, V, VI, and VII as having "pour objet le quadrilatère coupé par une transversale . . . " [7, p. 74]. Four points in a Euclidean plane, no three collinear, determine six straight lines, namely the sides of a **complete quadrangle.** If no two of the sides are parallel, then a seventh straight line

- may be parallel or not to one of the six sides;
- may pass through none, one, or two of the three intersection points of opposite sides.

That makes six cases, and Pappus takes up *five* of them in six lemmas, as in Table 4. Chasles draws our attention to those five cases severally, in consecutive paragraphs [7, pp. 74f.].

If Chasles does *not* also draw our attention to Lemma VIII as a case of the Hexagon Theorem, *this* leaves us with the open question of whether he saw

Table 4: Pappus's lemmas on complete quadrangles

	intersection points		
	none	one	two
parallel to a side	I, II	VII	VI
not parallel	IV		V

the connection himself.

7 Central projections

In Figure 5 and 28 for the fully intersecting case of Pappus's Theorem, each of four straight lines through A cuts respectively

- DC at C, L, K, and D;
- DB at O, B, F, and D.

In other words, the **central projection,** from A, of DC onto OB takes the **range** $CLKD$ to $OBFD$. In notation used by Robinson [44, p. 24] and Coxeter [9, p. 242],

$$CLKD \overset{A}{\underset{\wedge}{=}} OBFD; \tag{11}$$

however, Robinson would write the range $CLKD$ as (C, L, K, D).

Dillon [15, pp. 262–6] does without the notation of (11). Instead, she denotes a central projection from one straight line to another by a single letter, such as T, so that (11) might become

$$T(C, L, K, D) = (O, B, F, D).$$

We can identify the relation denoted by $\overset{A}{\underset{\wedge}{=}}$ in (11) with the function T. We might write the latter as T_A. All three authors refer to it as a *perspectivity.*

Still in Figure 5 and 28, projection from E takes $OBFD$ to $CBHM$, so that

$$OBFD \overset{E}{\underset{\wedge}{=}} CBHM. \tag{12}$$

Finally, if GH meets CD at a point K', not shown as such in Figure 5 and 28, then

$$CBHM \overset{G}{\underset{\wedge}{=}} CLK'D. \tag{13}$$

The only composite of central projections that fixes three points of a straight line will be the identity, by the *Fundamental Theorem of Projective Geometry,* restricted to an affine plane. When we have this theorem, in § 10, then (11), (12), and (13) give us that K' must be K, and thus the fully intersecting case of Pappus's Theorem holds.

In a projective plane, the Fundamental Theorem is a *consequence* of Pappus's Theorem – which, again, Hilbert called Pascal's Theorem. Desargues's Theorem is not also needed, at least not as an axiom, because it is a consequence of "Pascal's" Theorem, as Hessenberg shows in "Beweis des Desarguesschen Satzes aus dem Pascalschen," 1905 [25]:

> Daß der Pascalsche Satz aus dem Desarguesschen durch reine Verknüpfung nicht gefunden werden kann, hat Herr Hilbert in den „Grundlagen der Geometrie" gezeigt. Daß aber umgekehrt der Desarguessche Satz eine Folge des Pascalschen ist und durch reine Verknüpfung aus ihm hergestellt werden kann, scheint bisher nicht bemerkt worden zu sein. Wenigstens finden sich in allen den Gegenstand behandelnden Arbeiten Bemerkungen wie die, daß der projektive Fundamentalsatz aus dem Desarguesschen und Pascalschen Satz folgt, oder daß jeder ebene Schnittpunktsatz aus dem Pascalschen und Desarguesschen Satz bewiesen werden kann.
>
> That Pascal's theorem from Desargues's cannot be found, Mr Hilbert has written in *The Foundations of Geometry.* But that, conversely, Desargues's Theorem is a consequence of Pascal's, and can be proved from it by pure combination, seems to have gone unnoticed so far. At least all the works dealing with the subject contain remarks such as that the Fundamental

Theorem of Projective Geometry is derived from Desargues's and Pascal's Theorem, or that every plane intersection theorem can be proved from Pascal's and Desargues' Theorem.

Pappus's Theorem might not hold in a projective plane, depending on how one defines this.

We are working in an affine plane in which we already have the parallel case of Pappus's Theorem. We shall obtain from this the parallel case of Desargues's Theorem, by Hessenberg's method. This will allow us to use *Elements* VI.2, which is Thales's Theorem, not as a theorem, but as a definition. The fully intersecting case of Pappus's Theorem will follow in § 10.

8 Parallel Desargues

In an affine plane, in addition to central projections, there are **parallel projections.** Under one of these, the image of a range of points of one straight line is a range of *another* straight line, provided the straight lines joining the points of the former range with their images in the latter range are parallel to one another. One of these parallel straight lines, but at most one, may fail to exist uniquely, because the nominal endpoints are identical; it can then be ignored.

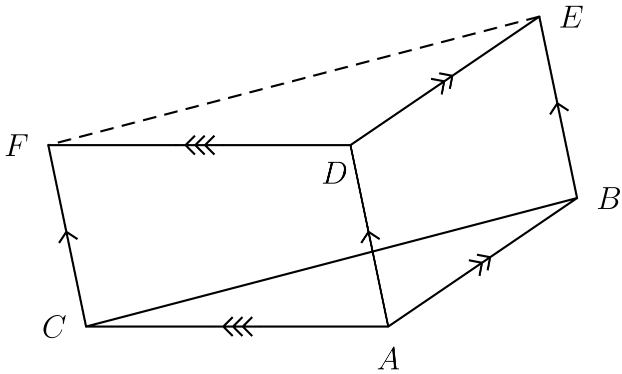

Figure 18: Desargues's Theorem, parallelogrammatic case

In § 9, we shall establish a that the only composite of parallel projections that fixes two points of a straight line is the identity. We shall do it by means of the parallel case of Desargues's Theorem. Now we prove this case.

As with the parallel case of Pappus's Theorem, so with Desargues's, we can understand it as two cases:

parallelogrammatic, as in Figure 18;

trapezoidal, as in Figure 8 and 21.

In Figure 18 and 21, one hypothesis is

$$AB \parallel DE, \qquad AC \parallel DF. \tag{14}$$

In the former diagram, we assume also

$$AD \parallel BE, \qquad\qquad AD \parallel CF. \tag{15}$$

Then

$$BE \parallel CF \tag{16}$$

by *Elements* I.30, the transitivity of parallelism. Now $ADEB$ and $ADFC$ are parallelograms, so we can use *Elements* I.33 and 34, as we did in § 5 for the parallelogrammatic case of Pappus's Theorem. We now have

$$AD = BE, \qquad\qquad AD = CF, \tag{17}$$

so

$$BE = CF, \tag{18}$$

and thus $BEFC$ is a parallelogram; in particular,

$$BC \parallel EF. \tag{19}$$

If we understand

- a range of two points as a **directed segment,** and
- the equations in (17) and (18) as equations of such things,

then the relation of equality here is precisely the transitive closure of the relation that holds between opposite sides of a parallelogram. This is the meaning of

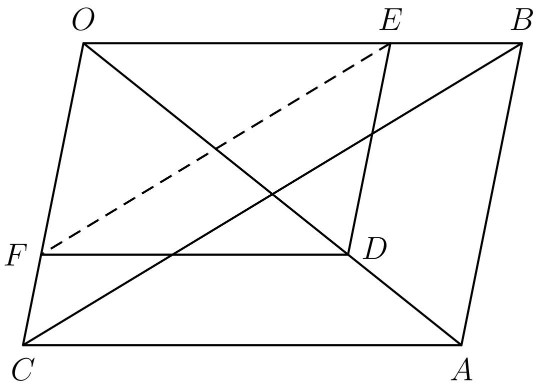

Figure 19: Degenerate trapezoidal case of Desargues

the parallelogrammatic case of Desargues's Theorem. We can understand the class of *all* directed segments that are equal to *one* of them as a **vector.**

Suppose again (14), but now, instead of (15), AD, BE, and CF have a common point O. We still want to show (19). Possibly $OBAC$ is a parallelogram, as in Figure 19. Here we have a straightforward exercise for Book 1 of the *Elements.* If we fill out the diagram as in Figure 20, then the complementary parallelograms BD and DC are equal, by Proposition 43. (In Greek, you know from their *gender* that BD and DC are parallelograms, not straight lines; see [38, § 3].) The tri-

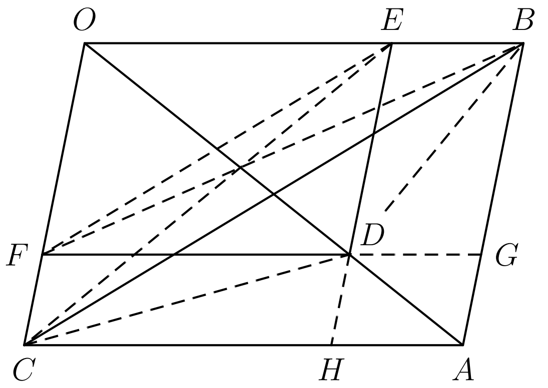

Figure 20: Proof

angles that are halves of the parallelograms are now equal, by Proposition 34 (as we discussed in § 2):

$$BED = FCD.$$

Applying Proposition 37 yields

$$BEF = FCE.$$

Therefore (19) holds, by Proposition 39.

We may now assume that OB is not parallel to AC. This is as in Figure 21, where pairs of solid, bold straight lines are parallel, either by our hypothesis (14) or by the construction that we now make. We let

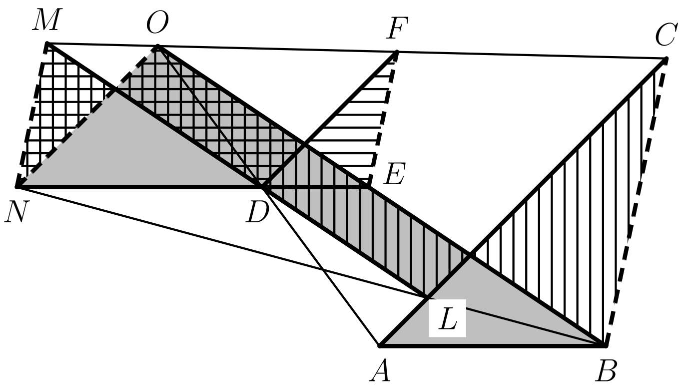

Figure 21: Hessenberg's proof of the trapezoidal case of Desargues's Theorem

the parallel to OB through D meet AC at L and OC at M. We also let BL meet DE at N. As Hessenberg shows [25], we can prove (19) with three applications of the parallel case of Pappus's Theorem. Applying this to hexagon $ONDLAB$, which is shaded, we obtain

$$ON \parallel AL. \tag{20}$$

Therefore, in hexagon $ONMLCB$, which is vertically scored,

$$BC \parallel MN. \tag{21}$$

Because, again, parallelism is transitive by *Elements*

1.30, (20) gives us

$$ON \parallel DF.$$

Hence, in hexagon $ONMDFE$, horizontally scored,

$$EF \parallel MN.$$

This and (21) give us (19), as desired.

9 Ratios

Given now two ranges OAB and OCD of *different* straight lines that intersect at O, let us write

$$OAB \approx OCD, \tag{22}$$

provided

$$AC \parallel BD,$$

as in Figure 8. The relation $\approx$ is not transitive, since by definition (22) never holds when OAB and OCD are on the *same* straight line. However, let us denote the transitive closure of $\approx$ by $\sim$. For every range OAB and every point D distinct from O, even if D lies on OB as in Figure 22, there is a point C on OD such that

$$OAB \sim OCD. \tag{23}$$

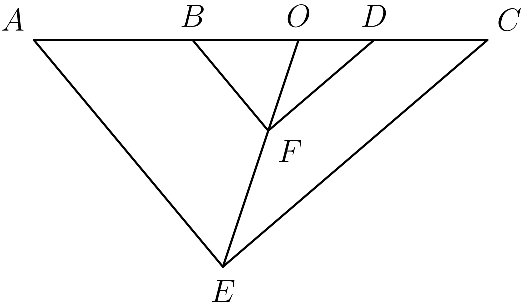

Figure 22: Two parallel projections

Moreover, the point C is unique. Indeed, if

$$OAB \approx OEF \approx OCD, \tag{24}$$

and (as in Figure 8) D is not on OB, then by parallel Desargues, (22) holds. Suppose now

$$OAB \approx OEF \approx OGH \approx OCD, \tag{25}$$

but H is on OB. Then D is not on OB.

1. If F is not on OD, then by parallel Desargues again, (24) holds, and then (22).
2. If F is on OD, as in Figure 23, then let L be on neither OB nor OD, and let

$$OAB \approx OKL.$$

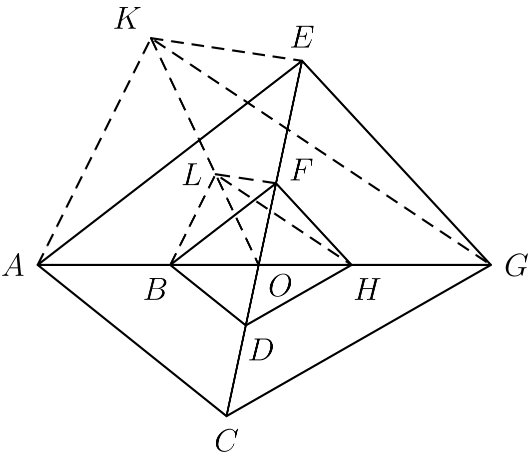

Figure 23: Web of projections

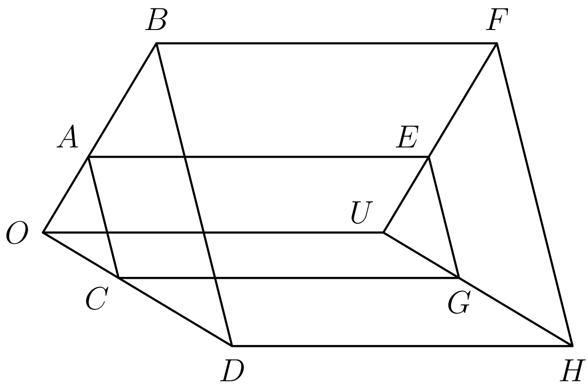

Figure 24: More projections

> By Desargues, we can replace OEF in (25) with OKL, so that we are in the previous case.

Repeated application of this yields that if (23) holds and D is not on OB, then (22) holds, so D is unique. Likewise, if (23) holds, but D is on OB, then (24) holds for some OEF. In particular, this is true when OCD is OGB for some G. Thus, if

$$OAB \sim OGB, \tag{26}$$

then A and G must be the same point.

Provided that, as in Figure 24, OAB and UEF are ranges of distinct *parallel* straight lines, and

$$OU \parallel AE, \qquad AE \parallel BF,$$

we may write

$$OAB \approx UEF.$$

If also (22) holds, and U is not on OD, then for some H distinct from U and some G on UH,

$$OCD \approx UGH, \qquad UGH \approx UEF.$$

Thus we have a kind of "commutativity." Thanks to this and parallelogrammatic Desargues, given a chain of ranges linked by the expanded relation $\approx$, we may assume that all but the last one or two ranges have the same initial point. If now we let $\sim$ be the transitive closure of the expanded relation $\approx$, we can still conclude from (26) that A and G are the same point. Thus we have the theorem, stated in § 8, that among composites of parallel projections, only the identity fixes two points of a straight line.

If now

$$ABC \sim DEF,$$

we can write this as either of

$$AB : AC :: DE : DF, \quad BA : AC :: ED : DF.$$

Instead of the equals sign (=), we use the sign $::$ here, as in (3), because what the sign is relating are *identical* equivalence classes, not geometrical figures (such as triangles or segments of straight lines) that may be distinct from one another, even while being equal. We can understand the ratio

- $AB : AC$ as the class of ranges related to ABC by $\sim$;
- $BA : AC$ as the class of ranges related to $AB'C$ by $\sim$, where B' is given by

$$B'A = AB.$$

We can understand AB, for example, either as a directed segment or as the vector it represents.

Generally,

$$BA : AC :: AB : CA.$$

We can define composition of ratios by the rule

$$(OA : OB) \cdot (OB : OC) :: (OA : OC).$$

Evidently, composition of ratios is associative. It is commutative, thanks to the parallel case of Pappus's Theorem. We can allow a neutral ratio, the "ratio of equality," $AB : AB$. If we exclude the ratios $AA : AB$

and $AB : AA$, then all of the other ratios compose an abelian group.

We can also allow *one* of the excluded ratios, treating it as zero, and define sums of ratios, so as to obtain a field.

10 Fully intersecting Pappus

First stated in § 4, Thales's Theorem is that, in a triangle OEF, a point A divides the side EF in the same ratio that a point D divides the side OF, just in case the straight line AD is parallel to the base OE. This is now true by definition, thanks to the work of § 9. Symbolically,

$$OE \parallel AD \iff EA : AF :: OD : DF.$$

We can write the proportion here as

$$1 :: (EA : AF) \cdot (FD : DO). \tag{27}$$

Suppose the two straight lines OE and AD are *not* parallel, but meet at C, as in each of the diagrams of Figure 25. Let the parallel to EF through O meet AD at B. Then by Thales's Theorem,

$$FD : DO :: AF : OB,$$

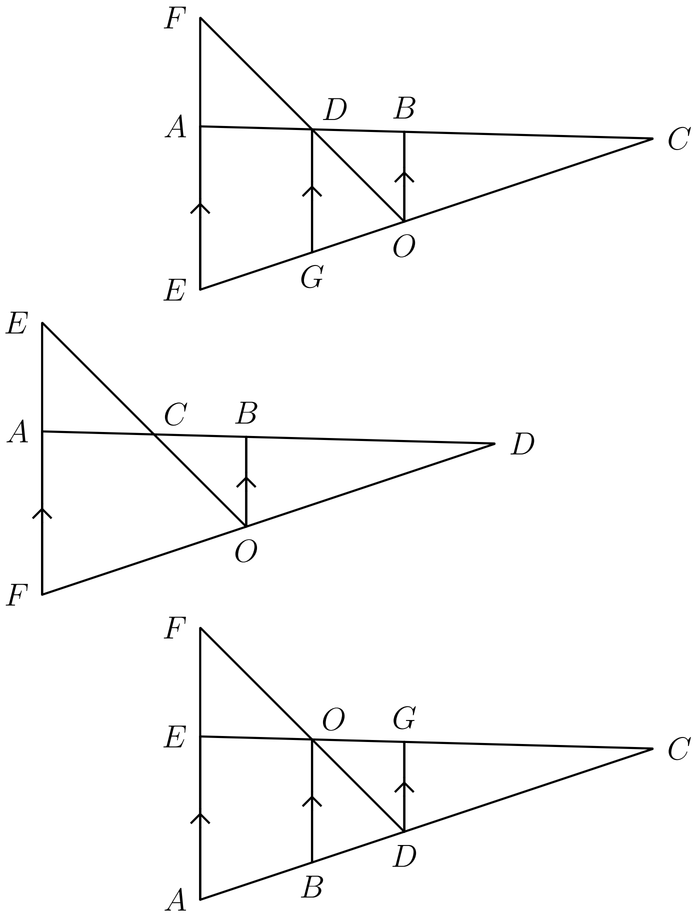

Figure 25: Theorem of Menelaus

where we should understand OB and the others as vectors, rather than just as directed segments, as suggested in § 9. Also

$$\begin{aligned} EC : OC &:: EA : OB \\ &:: (EA : AF) \cdot (AF : OB) \\ &:: (EA : AF) \cdot (FD : DO). \end{aligned}$$

This is the **Theorem of Menelaus** [43, p. 69, n. 84], [50, pp. 446–51]. We can understand (27) as a limiting case.

Similarly, still in Figure 25,

$$AE : AF :: (AE : OB) \cdot (OB : AF),$$

and therefore

$$AE : AF :: (AC : CB) \cdot (BD : DA). \tag{28}$$

The ratio $AE : AF$ depends only on A and the three straight lines through O. The same then is true of the compound ratio in (28). This compound is today the **cross ratio** of the four points A, B, C, and D, taken in that order. We may write (28) as

$$AE : AF :: (A, B; C, D). \tag{29}$$

Pappus gives a consequence in his Lemma III: in Figure 26, in our notation,

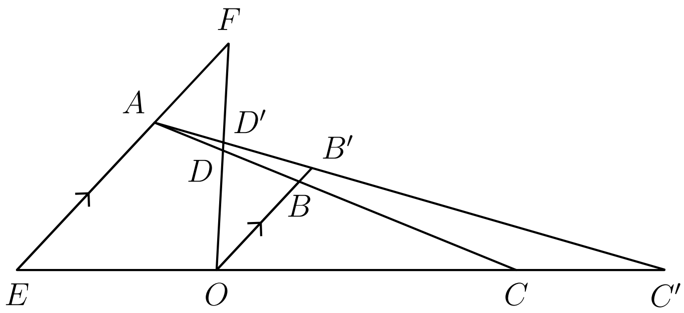

Figure 26: Lemma III of Pappus

$$(A, B; C, D) :: (A, B'; C', D'). \tag{30}$$

Meanwhile, comparison of the first two diagrams of Figure 25 shows

$$AF : AE :: (A, B; D, C);$$

of the third and first, where $DG \parallel AE$,

$$EA : EF :: (E, G; C, O),$$

and therefore, by Thales,

$$EA : EF :: (A, D; C, B).$$

The group of permutations of three objects being generated by two distinct interchanges, (29) must hold whenever the same permutation is applied to (B, C, D) on the right and (A, F, E) on the left. Moreover, the composites

$$(AC : CB) \cdot (BD : DA), \quad (CA : AD) \cdot (DB : BC),$$
$$(BD : DA) \cdot (AC : CB), \quad (DB : BC) \cdot (CA : AD)$$

are all the same, which means

$$\begin{aligned} &(A, B; C, D), \quad (C, D; A, B), \\ &(B, A; D, C), \quad (D, C; B, A), \end{aligned} \tag{31}$$

each starting with a different point, are all the same. Therefore (30) holds, whenever the same permutation is applied to the entries on either side.

As an application, in Figure 27, where

$$ABCD \underset{\wedge}{\overset{O}{=}} AEFG \underset{\wedge}{\overset{O}{=}} HKLG,$$

we have also

$$(A, B; C, D) :: (A, E; F, G) :: (H, K; L, G).$$

Thus, in Figure 5 and 28, where we have the projections from A and E given by (11) and (12) in § 7, we can conclude

$$(C, L; K, D) :: (O, B; F, D), \tag{32}$$
$$(O, B; F, D) :: (C, B; H, M), \tag{33}$$

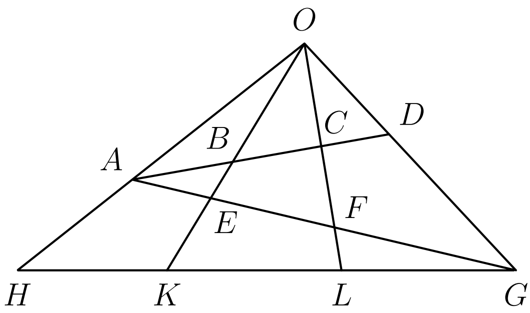

Figure 27: Lemma III of Pappus, generalized

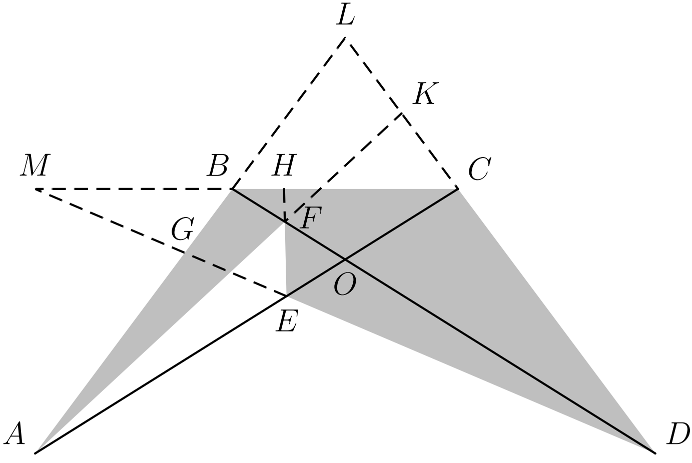

Figure 28: Pappus's Theorem, fully intersecting case

and then

$$(C, L; K, D) :: (C, B; H, M). \tag{34}$$

Consequently, the intersection of LB and DM being G, we must have

$$CLKD \overset{G}{\underset{\wedge}{=}} CBHM, \tag{35}$$

and in particular the straight line passing through G and H passes also through K. As noted in § 7, this is by the Fundamental Theorem of Projective Geometry in an affine plane. For Pappus, that (34) entails (35) is his Lemma x, the converse of Lemma III. The point is that (13) (back in § 7) gives us

$$(B, C; H, M) :: (C, L; K', D),$$

where K' is the intersection of GH and CD. This with (34) gives us

$$(C, L; K, D) :: (C, L; K', D),$$

and this means

$$CK : KL :: CK' : K'L,$$

so K and K' must be identical.

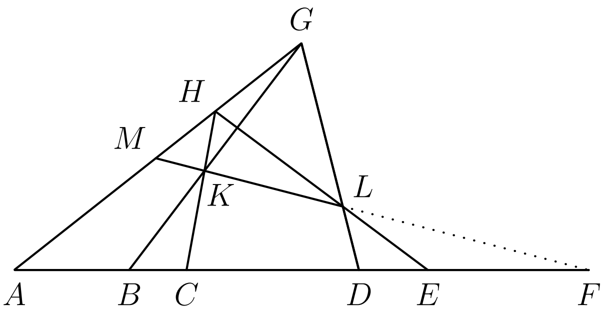

Figure 29: Lemma IV of Pappus

We now have Pappus's Theorem in the fully intersecting case; this is his Lemma XIII.

We can apply the kind of argument that we have just used to Figure 29, where $GHKL$ is a complete quadrangle, as discussed in § 6. Five of the six sides cut a straight line at A, B, C, D, and E, and the sixth side, KL, cuts AG at M. If KL cuts AE at F, then

$$ABCF \underset{\wedge}{\overset{K}{=}} AGHM \underset{\wedge}{\overset{L}{=}} ADEF,$$

and therefore

$$(A, B; C, F) :: (A, D; E, F).$$

Simson gives such an argument [52, p. 71], as Jones points out [33, pp. 558f.]. Pappus's Lemma IV is the converse, but his proof relies on the converse of Menelaus's Theorem and other auxiliary straight lines, not KM.

11 Remaining cases of Pappus

In § 10, Pappus's Lemma III gave us that, when four straight lines share a point, the cross ratio of the points where a straight line not through the common point cuts the four straight lines is always the same. We obtained this sameness as an application of (28), written also as (29), with points and straight lines as in Figure 25. The proportion (28) itself is Pappus's Lemma XI, and we may write it as

$$(A, \infty; E, F) :: (A, B; C, D).$$

Here we are hinting at the projective plane that we said we were not actually going to be working in. Given the identity of the cross ratios of (31), we may find it convenient to write the ratio $AE : AF$ in any of the forms

$$\begin{aligned} &(A, \infty; E, F), && (E, F; A, \infty), \\ &(\infty, A; F, E), && (F, E; \infty, A). \end{aligned}$$

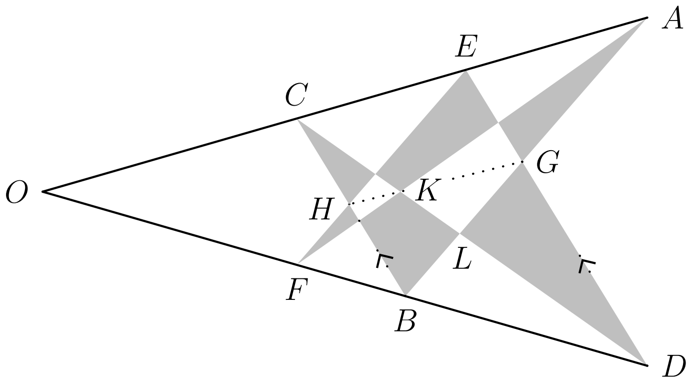

Figure 30: Pappus's Theorem, intersecting case $(1,1)$

If we use Figure 13 in place of 5 or 28, so that AE and DB are now parallel, then (32) and (33) become

$$\begin{aligned}(C,L;K,D) &:: (\infty,B;F,D),\\ (\infty,B;F,D) &:: (C,B;H,M),\end{aligned} \tag{36}$$

and (34) still holds. This is Pappus's Lemma XII, which is intersecting case $(0,2)$ of Pappus's Theorem.

In a diagram for the fully intersecting case of Pappus's Theorem, as in Figure 5, it doesn't matter whether $CD \parallel EF$. If it is, then as in Figure 30, we obtain a figure for intersecting case $(1,1)$. Here, we still have (32), but (33) becomes

$$(O, B; F, D) :: (C, B; H, \infty),$$

so that (34) becomes

$$(C, L; K, D) :: (C, B; H, \infty). \tag{37}$$

This would also be true, if KG passed through H, and therefore it does pass – this result is Pappus's Lemma XIV.

In Figure 30, we already happen to know that G, H, and K are on a straight line, because BC and FA intersect. However, this alone does not establish what we want, since possibly three non-adjacent, non-opposite sides of the hexagon are parallel, as in Figure 31.

Likewise, we are forced into intersecting case $(0, 1)$ by Figure 32, where (32) and (33) become (36) and

$$(\infty, B; F, D) :: (C, B; H, \infty).$$

Again (37) follows, so KG passes through H. (Also, we could have used a relabelled Figure 13, assuming $CD \parallel EF$ there.)

We can be forced into intersecting cases $(1, 0)$ and $(0, 0)$ if the points of our straight lines are not ordered. For example, over a field in which ζ is a primitive cube root of unity, or over a field of characteristic

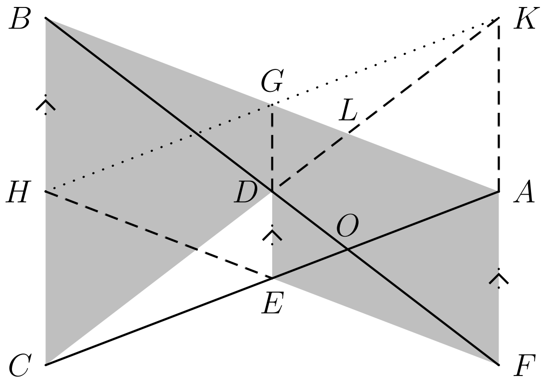

Figure 31: Pappus's Theorem, intersecting case $(1, 1)$

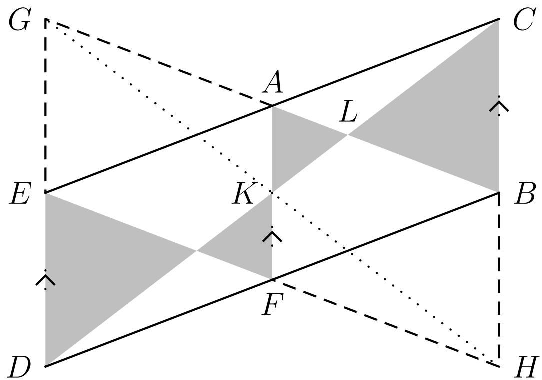

Figure 32: Pappus's Theorem, intersecting case $(0, 1)$

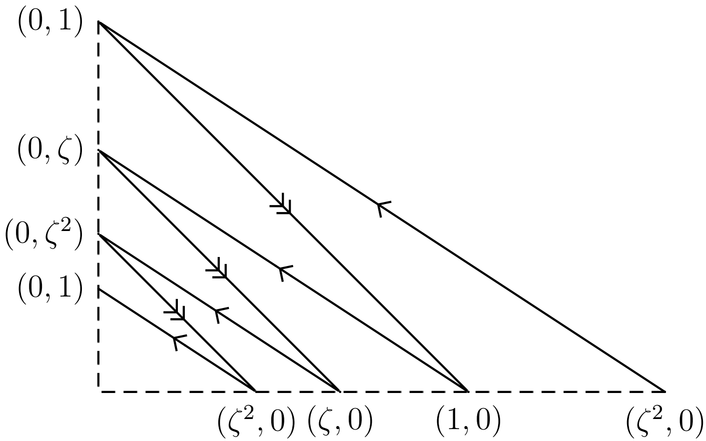

Figure 33: Hexagon requiring case $(1, 0)$

3, we may be given the hexagon of Figure 33 or 34. For proofs, we can use Figure 35 and 36, where we obtain (34) in the form

$$(C, \infty; K, D) : : (C, B; H, \infty),$$

which means

$$CK : CD : : CH : BH.$$

Since $CDGB$ is a parallelogram, the last proportion is also

$$CK : BG : : CH : BH.$$

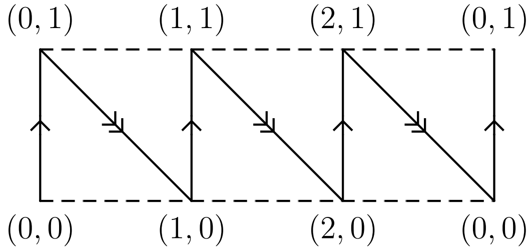

Figure 34: Hexagon requiring case $(0, 0)$

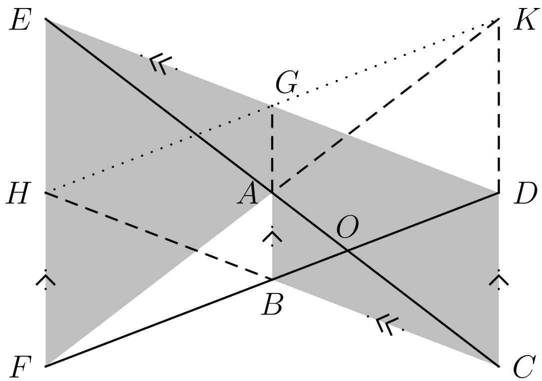

Figure 35: Pappus's Theorem, intersecting case $(1, 0)$

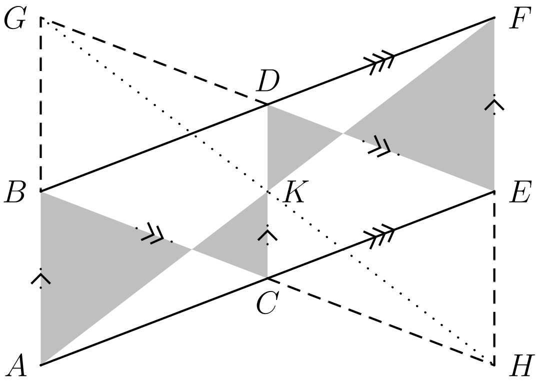

Figure 36: Pappus's Theorem, intersecting case $(0, 0)$

By Thales, H must lies on GK.

In the mixed case, we still suppose AB and DE intersect at G, but CD and FA are parallel. By the parallel case of Pappus's Theorem, BC and EF must not be parallel, so they intersect at H. Also, CD and AB must still intersect at L. If BC and DE still intersect at M, as in Figure 37 and 38, then (34) becomes

$$(C, L; \infty, D) :: (C, B; H, M). \tag{38}$$

In particular, this ratio cannot be $(C, L; K, D)$ for any K on CD. It *would* be this ratio if K were where GH cut CD. Therefore those two straight lines are parallel.

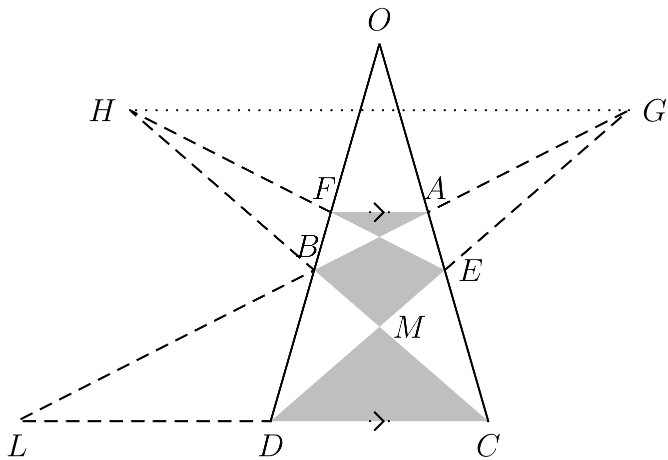

Figure 37: Pappus's Theorem, mixed case $(1, 2)$

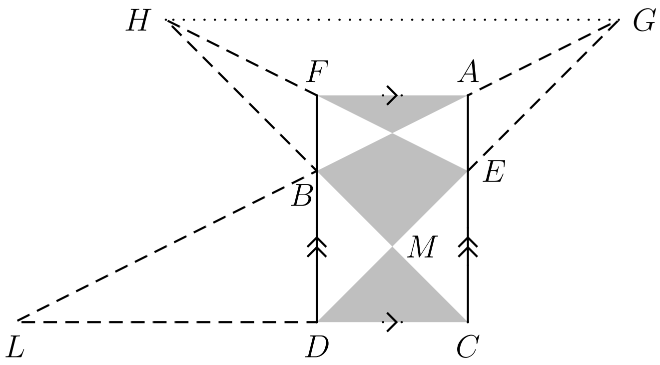

Figure 38: Pappus's Theorem, mixed case $(0, 2)$

If instead BC and DE are parallel, as in Figure 39 and 40, then (38) becomes

$$(C, L; \infty, D) :: (C, B; H, \infty),$$

whence

$$(C, L; \infty, D) :: (B, C; \infty, H),$$
$$(L, \infty; D, C) :: (C, \infty; H, B),$$

that is,

$$LD : LC :: CH : CB.$$

By Thales,

$$LD : LC :: DG : CB.$$

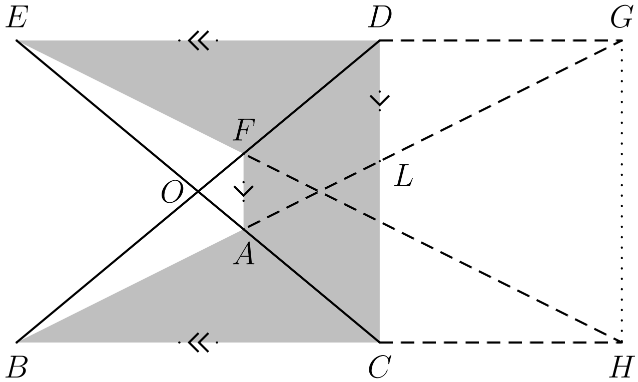

Figure 39: Pappus's Theorem, mixed case $(1, 1)$

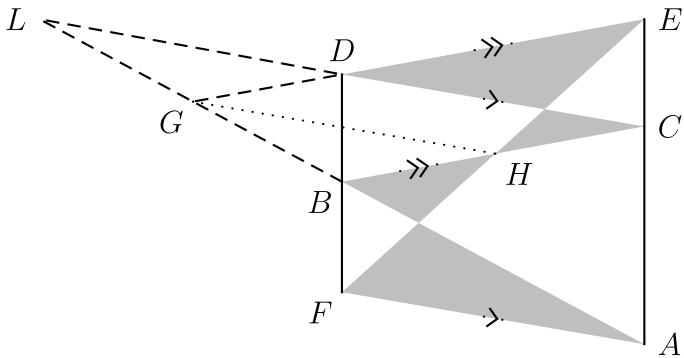

Figure 40: Pappus's Theorem, mixed case $(0, 1)$

Thus, $CH = DG$, and $CDGH$ is a parallelogram.

12 Conclusion

We have established Pappus's Hexagon Theorem in all affine cases, using only affine results from Book I of Euclid's *Elements*:

- no comparisons of non-parallel segments,
- no Archimedean postulate,
- no ordering of points on straight lines.

In ancient fashion, we have used diagrams to show which points are collinear. We have used some modern notational conveniences, particularly by using ∞ as a place-holder. We also used the notion of a symmetry group. In principle, we could have avoided all of this by writing more out in words.

Beyond diagrams, Pappus has his own conveniences, at least in the possibility of omitting words. His Lemma III, whose conclusion is expressed by us in (30) with letters as in 26, or with his letters as

$$(\mathit{\Theta}, Z; E, H) :: (\mathit{\Theta}, \mathit{\Gamma}; B, \mathit{\Delta}),$$

is originally as follows, in Jones's edition [33, p. 263]:

εἰς τρεῖς εὐθείας τὰς ΑΒ, ΓΑ, ΔΑ διήχθωσαν δύο εὐθεῖαι

> αἱ ΘΕ, ΘΔ. ὅτι ἐστὶν ὡς τὸ ὑπὸ ΘΕ, ΗΖ πρὸς τὸ ὑπὸ ΘΗ, ΖΕ, οὕτως τὸ ὑπὸ ΘΒ, ΔΓ πρὸς τὸ ὑπὸ ΘΔ, ΒΓ·

Here is a literal translation, with words underlined that are only implicit in the Greek:

> To the three straight lines AB, ΓA, and ΔA have been drawn the two straight lines ΘE and $\Theta\Delta$. I say that it is the case that as the rectangle contained by ΘE and HZ is to the rectangle contained by ΘH and ZE, so the rectangle contained by ΘB and $\Delta\Gamma$ is to the rectangle contained by $\Theta\Delta$ and $B\Gamma$.

Here the third-person perfect passive verb "have been drawn" is to be understood as an imperative, or as a subjunctive used imperatively, like the verbs in "So be it" and "God bless you." We would normally say "*let* have been drawn." Actually, we would say "let *be* drawn," but the third-person passive imperative verb διήχθωσαν that we are translating is in the perfect aspect. In any case, Pappus's language is economical.

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