Numbers and Notes in Plato and Euclid

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Two Scales

For the Greeks the natural succession was Orpheus descending;
Gold into Iron; Phlegethon, ground of passion;
Post and lintel of tragic song.
Our scale rises, Gothic vaultings,
The grave that moults the angelic butterfly:
The third is love, the seventh pain, the diapason fire.

(Charles Greenleaf Bell [3, p. 28])

1 Introduction

"Why do so many people positively dislike mathematics?" Timothy Gowers raises that question in *Mathematics: A Very Short Introduction* [19, pages 131–3]. Part of the answer is the way our subject is cumulative. In particular:

> Every so often, a new idea is introduced which is very important and markedly more sophisticated than those that have come before, and each one provides an opportunity to fall behind. An obvious example is the use of letters to stand for numbers . . .

There are other things that letters stand for. These include musical pitches or notes. Can music then be disliked the way mathematics is?

Music would seem to exist to be *heard.* Those who cannot listen may still *see* or *feel* music – as at a performance of Sweet Honey in the Rock, the a capella ensemble who include a sign language interpreter.

Some people can read a musical *score*; for others, it may be as opaque as a polynomial equation, if not more so.

In an equation such as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

not only do the letters represent numbers – usually so-called *real* numbers. The letters do this in two different ways, depending on whether they come from the beginning or end of the alphabet.

On a musical *staff,* each of the four gaps between the five horizontal lines represents an *interval* that is either a *major third* or *minor third.* Which one depends not only on which gap, but also on which *clef* appears at the left end of the staff. There are additional thirds above and below the staff. The patterns are as in Table 1, in the notation of Thomas Hillen in "A Classification of Musical Scales Using Binary Sequences" [21], where 0 stands for the minor third, 1 for the major.

None of that last paragraph may make any sense,

Table 1: Thirds on staves

treble	alto	base
1	1	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	0

at *any* level. I hope to make some *mathematical* sense of such things, with the help of Plato and Euclid.

If I think about my youth, and what could have been different then, I may regret not getting a better education in music. This is at least partially my fault. Still, I don't know who could have told me (if only I had asked) the things that I want to work out here.

That does *not* mean I wish I could have had what Gowers imagines in a recent essay, "Thoughts on the Leiden Declaration" [20]:

> what reason is there to suppose that ChatGPT 8.2 wouldn't be able to have a short interaction with you about your mathematical tastes and background and then write the ideal textbook just for you?

In geometry class, instead of our contemporary textbook, which illustrated the concept of congruence with a photograph of a machine stamping out foil trays for TV dinners [39, page 13], I didn't want "the ideal textbook just for me"; I wanted to go back to the *human* foundations. I wished we could be reading the *Elements* of Euclid [14, 16, 17] – as I ended up doing in college [32].

In the Introduction of his *Ancient Greek Music,* M. L. West has a nice exposition of "Musical Preliminaries" for the beginner [40, pages 8–12]; however, it starts with a piano keyboard. We are going start with a simpler instrument, which has a single string. You can make a version of it by stretching a rubber band around a box.

Meanwhile, since we have already brought up some technical terms, let us give a few more hints about them now. A musical *interval* will determine a ratio. Each of the intervals looked at above is called a *third,* because it is associated with *three* things, such as two adjacent horizontal lines of a staff, along with the space between. A sequence of seven major and minor thirds is one way to determine a *scale,* as Hillen observes [21, page 119]. There must then be exactly three major thirds in the sequence [21, Lemma 1, page 122], so the sequences in Table 1 can be filled out in

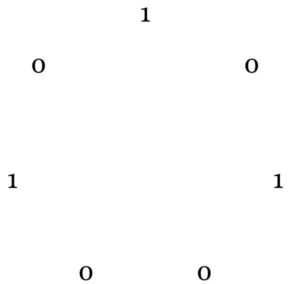

Figure 1: Circle of thirds

only one way (in either direction). Hillen does not discuss the ratios that the thirds determine, but today they are commonly $\sqrt[3]{2}$ for the major third, $\sqrt[4]{2}$ for the minor.

Adding intervals end to end corresponds to multiplying their ratios, so that one of the sequences of thirds just described has the ratio of 4. The endpoints of each interval are *pitches* or *notes,* but each of these is identified with the one that makes with it an interval whose ratio is 2. That interval is an *octave,* because seven pitches divide it into eight parts. Each of the sequences in Table 1 loops around as in Figure 1.

Intervals as ratios are a topic of Pair *et al.* in "An exploration of mathematical and musical connections" [30]. Not mentioning thirds explicitly, they do note that the *fourth* and *fifth* have modern values $2^{5/12}$ and $2^{7/12}$, ancient values $4/3$ and $3/2$. These "Pythagorean" values are the ones that will interest us. They will give rise to the pattern in Figure 1, and ultimately that of Figure 17 in § 7, depicting a musical scale of 53 notes.

That scale is not the *theoretical* ultimate. In present terms, no scale is ultimate, because, while obviously

$$(2^{7/12})^{12} = 2^7, \qquad (2^{31/53})^{53} = 2^{31},$$

no power of $3/2$ (unless we count the zeroth) is exactly a power of 2; nonetheless,

$$\begin{aligned} 3^{12} &= 531\,441, & 2^{19} &= 524\,288, \\ 3^{53} &\approx 1.938 \times 10^{25}, & 2^{84} &\approx 1.934 \times 10^{25}. \end{aligned} \tag{1}$$

I am trusting a calculator for the approximate values of the higher powers of two and three there; the values of 2^{18} and 3^{12} are computed in the *Sectio Canonis* [1, 13, 38, Proposition 9], which is attributed to Euclid. For the general assertion, I trust the logic of what is called Euclid's Lemma and is Proposition 30 of Book VII of the *Elements*.

The sections below are as follows.

2. In the *Sectio Canonis,* there is a practical way of marking out lengths of the aforementioned string, for harmonious plucking or strumming.

3. Today, numbers are usually real numbers, and a ratio of two of them is another one of them. A magnitude then is a number of units, such as meters or seconds. Two magnitudes of different kinds have a ratio, which is another magnitude, such as a number of meters-per-second. For Euclid, a number is what we call a counting number or a positive integer; strictly, it is such a number of units. Magnitudes are generally lengths, areas, or volumes, but they are *not* numbers of units. A ratio is an abstraction, a new kind of thing. Magnitudes of the same kind have a ratio, by the definition in Book v of the *Elements*; numbers have a ratio by the definition in Book vii. What really gets defined is when two ratios are the *same.* Although Euclid may not spell out details that we might wish to see today, the numerical definition of sameness of ratio applies to *commensurable* magnitudes, and it entails the general definition. Implicitly, two numbers have the same ratio as two other numbers when the *Euclidean Algorithm* has the same steps in either case.

4. In diagrams, line segments represent

- numbers in the *Elements,*

- musical pitches in the *Sectio Canonis.*

This is perhaps not an accident. Drawing numbers as line segments also helps us see the non-triviality of the commutativity of multiplication of numbers, which is Proposition 16 of Book VII of the *Elements.* Once we have this, commensurable magnitudes that have the same ratio by the general definition also do by the numerical definition. That general definition ensures that ratios are real numbers according to Dedekind's definition of those.

5. The *Sectio Canonis* presents also a *theoretical* way of marking out lengths on our musical string. If the only concordant ratios are multiple and *part-again,* then fourth, fifth, and octave must have the ratios $4 : 3$, $3 : 2$, and $2 : 1$.

6. Writing before Euclid, Plato in the *Timaeus* [34, 35] suggests how to find seven-note scales in the sequences of powers of of two and three.

7. Those sequences give us scales in which the numbers of notes themselves form the infinite sequence 2, 3, 5, 7, 12, 17, 29, 41, 53, . . .

8. That sequence is the sequence of denominators of the *convergents* and *semi-convergents* or *intermediate fractions* of $\log 3/\log 2$. This is all part of the modern theory of the Euclidean Algorithm, which we review for how it illuminates our scales.

2 Practical tuning

According to Andrew Barker in *The Science of Harmonics in Classical Greece* [2, pages 7–10]:

> from the perspective of most Greek theorists . . . the status of a sequence of notes as a genuine melody depends on its being rooted in a scale or attunement which is itself formed in a properly musical way . . . The task of harmonics, then, is to identify the structures on which melodies must be based . . .

We would seem to be given one of those structures in the work of Euclid called *Sectio Canonis.*

According to Barker in *Greek Musical Writings* [1, p. 190], this "little treatise" is

> attributed to Euclid in the manuscripts, and by Porphyry, who quotes from it at length . . . Parts of the treatise are also quoted by Boethius . . . If Euclid is indeed the author, the work will have been written around 300 B.C., but the attribution has been debated . . . There are no good reasons, however, for denying Euclid's authorship of the main part of the treatise, at least as much as Porphyry quotes . . .

Apparently Porphyry quotes the first sixteen of the twenty propositions. According to David Fowler in *The Mathematics of Plato's Academy*, where several pages are devoted to the *Sectio Canonis,* [18, pp. 138–46],

> Its author is unknown and its date is uncertain, but it clearly belongs within a mathematical, as contrasted with empirical, study of music, and this places it broadly within the Pythagorean and Academic traditions, rather than those associated with Aristotle's Lyceum.

The Latin title of the *Sectio Canonis* translates the Greek title, Κατατομὴ κανόνος. (I try to assume no knowledge of music here, but I do hope the reader recognizes the Greek letters, if only from their use in mathematics.) A *canon* or κανών is anything straight. It could be a rule: the literal kind that you draw straight lines with, or the metaphorical kind that you obey. In the present context, a canon is the instrument that we mentioned: a taut string that you can pluck or strike to make a sound. That sound is φθογγή in Greek, from the verb φθέγγομαι "raise one's voice"; in English, we see the noun in "diphthong."

Hold down the string of the canon, somewhere along its length (as you might hold a guitar string against a fret on the neck). You may also hold up the string with a bridge. If you now pluck one of the two parts of the string, you get a sound of some *pitch,* which will be *higher* than the pitch of the whole string. The two pitches may or may not sound good together, whether they come simultaneously (if you actually have two canons to work with) or sequentially. Instead of pitch, we may speak of the *note* of the string, though this word may suggest the sustaining of a pitch for a certain time. In any case, here is what Euclid says, *quâ* author of the *Sectio Canonis,* with Barker's translation (and my additions of transliterations of the Greek words, in their dictionary forms, that I have bolded in the original):

> Γινώσκομεν δὲ καὶ τῶν **φθόγγων** τοὺς μὲν **συμφώνους** ὄντας. τοὺς δὲ **διαφώνους,** καὶ τοὺς μὲν συμφώνους μίαν **κρᾶσιν** τὴν ἐξ ἀμφοῖν ποιοῦντας, τοὺς δὲ διαφώνους οὔ.
>
> Among notes (*phthongos*) we also recognize some as concordant (*symphônos*), others as discordant (*diaphônos*), the concordant making a single blend (*krasis*) out of the two, while the discordant do not.

The *Sectio Canonis* aims to find the lengths of string that are concordant. This will happen in two ways: by geometrical construction, based on experience of what *does* sound good, and by *a priori* axiomatic reasoning.

Some concordant lengths are marked out in the last two of the twenty propositions of the *Sectio Canonis.* The endpoints of the lengths are defined in alphabetical order, as in Figure 2:

1. The string of the canon is AB.
2. The string is divided into four equal parts at Γ, Δ, and E.

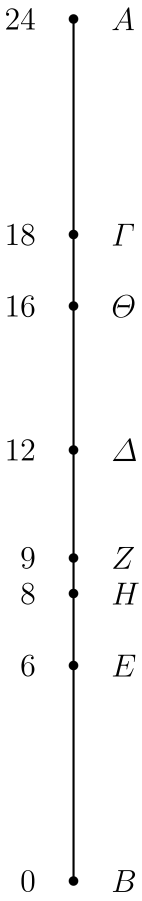

Figure 2: First sectioning of the monochord

3. The midpoint of ΓB is Z.
4. A third of ΔB is ΔH.
5. The remainder BH is equal to $H\Theta$.

Relative distances from B are as indicated in the diagram. Ratios of distances from consecutive points to B are either $4:3$ or $9:8$. In each of the four intervals where the ratio is $4:3$, two more points are added, as in Figure 3, where we have scaled the relative distances by $2^5 \cdot 3$, that is, 96. The labels of the new points are nearest the line; of the points not needed for their construction, furthest. The construction itself is as follows.

6. A third of ΘB is ΘK.
7. Equal to BK is $K\Lambda$.
8. An eighth of BE is equal to EM.
9. An eighth of BM is equal to MN.
10. A third of BN is equal to $N\Xi$.
11. Half of $B\Xi$ is equal to ΞO.
12. ΞO is equal to $O\Pi$.
13. A fourth of ΓB is ΓP.

Holding it at one of our points, if we pluck the string below the point, we get a note. Doing the same at another point, we have an **interval** between the notes. This interval somehow corresponds to the ratio of the distances of the points from B. An interval corresponding to $2:1$ is an **octave,** because there are ul-

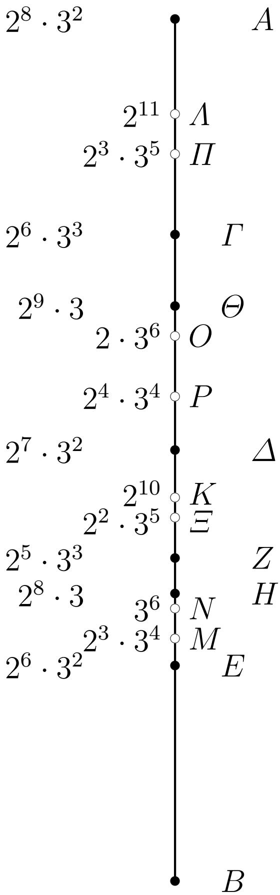

Figure 3: Ultimate sectioning of the monochord

timately eight points between A and $\mathit{\Delta}$ inclusive, as there are between $\mathit{\Delta}$ and E – and there is a *reason* for the number of points to be eight, as we shall see. In the *Sectio Canonis,* Euclid refers to the interval in question as a *diapasôn* (τὸ διὰ πασῶν διάστημα "the through-all interval"), as for example in Proposition 10, which we shall look at more below.

An inverval corresponding to $4 : 3$ is a **fourth,** or a **tetrachord,** because for example there are ultimately four points between A and $\mathit{\Gamma}$ inclusive. With one more point, as between A and $\mathit{\Theta}$, we have a **fifth,** whose ratio is $3 : 2$. For Euclid, in Proposition 11, the terms are *diatessarôn* (τὸ διὰ τεσσάρων διάστημα "the through-four interval") and *diapente* (τὸ διὰ πέντε διάστημα "the through-five interval").

The excess of the fifth over the fourth corresponds to the compound or product $(3 : 2)(3 : 4)$, which is $9 : 8$. The interval itself is a **tone** (ὁ τόνος, as in Proposition 16), and it is found between $\mathit{\Gamma}$ and $\mathit{\Theta}$. The tone is also a **major second.** It is a second, because there is no third point between its endpoints; major, because there is also a **minor second,** represented for example by $\mathit{\Lambda}$ and $\mathit{\Pi}$; the ratio here is $256 : 243$. Such an interval is a *limma* or **leimma** [1, page 222], from λεῖμμα, -ατος "remnant," and then each of the intervals represented by consecutive points (except B)

in Figure 3 is a tone or leimma.

The leimma is not quite half a tone. As Euclid points out, six tones together exceed 2, since

$$9^6 = 3^{12} = 531\,441, \qquad 8^6 = 2^{18} = 262\,144, \tag{2}$$

and twice the latter is less than the former, as in (1). The octave being a compound of two fourths and a tone, each fourth is less than two tones and a half; but it is two tones and a leimma.

We made our "section of the canon" in two stages. Our staging is *not* reflected in the text of the *Sectio Canonis*; there, Proposition 19 gets the reader through point Λ. Nonetheless, by constructing first the six points from A through Θ, then the eight from K to P, we illustrate remarks of Barker, which start by mentioning a student of Aristotle [2, pages 12 & 13]:

> The system within which most of Aristoxenus' simpler constructions find a place is a scale spanning two octaves . . . the principal sub-structures are the tetrachords, groups of four notes of which the outermost are a perfect fourth apart . . . Some of them are linked in 'conjunction' (*synaphē*), where the highest note of the lower tetrachord is also the lowest note of the tetrachord above. Others are in 'disjunction' (*diazeuxis*); that is, there is an interval (but no note) between them, and this interval is always a tone (in modern parlance, a major second) . . . chords and the

tones of disjunction are 'fixed' notes . . . The two notes inside each tetrachord, by contrast, are 'moveable'.

In our diagram, being further away from the point B at the bottom, the points that are higher up determine the lower notes. Something about a lower note that is mathematical lower is its *frequency.* Euclid would seem to be describing this in the introduction of the *Sectio Canonis,* here with Barker's translation:

> . . . ἀναγκαῖον τοὺς μὲν ὀξυτέρους εἶναι, ἐπείπερ ἐκ πυκνοτέρων καὶ πλειόνων σύγκεινται κινήσεων, τοὺς δὲ βαρυτέρους, ἐπείπξες ἐξ ἀραιοτέρων καὶ ἐλασσόνων σύγκεινται κινήσεων.
>
> . . . it follows that some notes must be higher, since they are composed of closer packed and more numerous movements, and others lower, since they are composed of movements more widely spaced and less numerous.

How we would measure the frequency of those movements is not clear. Euclid may assume that frequency is inversely proportional to the length of string being plucked. However, in his diagrams, notes are represented as line segments, and these seem to correspond to lengths of string.

For example, Figure 4 gives the diagram for Proposition 10, which begins as follows.

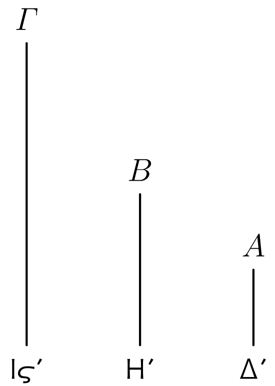

Figure 4: Proposition 10 of the *Sectio Canonis*

Τὸ διὰ πασῶν διάστημά ἐστι πολλαπλάσιον.

The octave interval is multiple.

ἔστω γὰρ νήτη μὲν ὑπερβολαίων ὃ A, μέση δὲ ὃ B, προσλαμβανόμενος δὲ ὁ Γ.

Let A be *nētē hyperbolaiōn,* let B be *mesē,* and let Γ be *proslambanomenos.*

Except that his letters for the notes are not A, B, and Γ, but A, B, and C, the translation is by Barker, who explains the Greek terminology in a footnote:

Nētē hyperbolaiōn is the highest note of the two-octave system within which Greek harmonic analysis

> was standardly set. *Mesē* is an octave lower, and an octave above *proslambanomenos,* the lowest note of the system. The author assumes familiarity with the named notes; he assumes that we know in advance of mathematical analysis the sizes of the intervals between them (in terms of octaves, fifths, fourths, tones, etc.); and he assumes that we know which of these intervals are concordant and which are not.

At this point in the *Sectio Canonis,* the reader does *not* officially know that the ratio of the octave is $2 : 1$. Nonetheless, the Greek numerals in the diagram are as given by Jan and by Menge, and they correspond to our 16, 8, and 4.

3 Euclidean ratios

We have been writing ratios as if, for any positive integers a and b that are prime to one another, the ratio of ac to bc (where c is a positive integer) is $a : b$. Our practice is based on what Euclid gives us in the *Elements.*

We have also suggested that $a : b$ might be understood as a single number, here a *rational* number, a/b. Other ratios may be irrational numbers, such as $\sqrt[3]{2}$. For Euclid, a ratio is only an abstraction from two positive integers or two *magnitudes.* As an identity or sameness of ratios, a proportion then is a relation among four magnitudes.

Euclid develops a theory of ratio and proportion

- for magnitudes in books v and vi of the *Elements*;
- for positive integers in particular, in books vii–ix.

The two theories are related in Book x. In the diagrams of books v and vii–ix, magnitudes and positive integers – *counting* numbers – are drawn as line segments, just as notes are drawn in the *Sectio Canonis.*

We shall review the theory of ratio and proportion, as needed for what Euclid gives us in the *Sectio Canonis.*

As Euclid puts it at the head of Book v (with my translation),

> Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.
>
> A ratio is, of two homogeneous magnitudes, a sort of bearing according to size.

Instead of "bearing" for Euclid's *schesis* (which is derived from ἔχω "to have or hold"), Heath has "relation." However, in mathematics at least, today we usually understand a relation as something that is either present or not. For example, this could be,

- for integers, being congruent *modulo* twelve;
- for notes, being concordant.

The big question for us now is when the ratio of two magnitudes is the *same* as that of another two.

Euclid does not spell out what it means for a magnitude to be homogeneous (*homogenēs*) or (as Heath puts it) of the same kind. Implicitly, there are three basic kinds of magnitudes: lengths, areas, and volumes. There might be others; for example, at the end of Book vi, Proposition 33 is that arcs that are cut from the circumference of a circle are in the ratio of the angles that cut them off, whether the vertices of the angles are at the center or on the circumference.

No ratio is ever contemplated of the circumferences of different circles. They are *not* shown to be in the ratio of the diameters, even though Proposition 2 of Book xii is that the circles themselves, or what we might call disks, are in the ratio of the squares on the diameters.

Possibly Proposition 33 was tacked on later to the end of Book vi. I suspect the same of Proposition 39 at the end of Book vii, because it contradicts what I said above about how a ratio is an abstraction. I shall come back to this.

The simplest possible relation between two magnitudes is that of equality. We used this relation in the constructions of figures 2 and 3.

There seems to be a modern prejudice whereby every equality can be considered as a sameness. Thus Barry Mazur suggests in a lecture [26, page 10] (delivered at my college, more than two decades after my graduation),

> Thanks to . . . the ability we have . . . of drawing a circle with any center and any radius – we can begin to construct many line segments that – in Euclid's terms – are "equal," (meaning: are of equal length).

I would say rather that segments as such may be equal, and when they are equal, they have the *same* length.

Lengths may be given to us in the form of a *graduated* ruler. With such a ruler, we can check whether two segments have the same length. However, we can compare two segments without measuring them; we can just lay one along the other. (The construction for this is Proposition 3 of Book 1 of the *Elements.*) Either one of the segments is greater, or else they are equal.

Like the ratio of two segments, the length of one of them is an abstraction. If we want, we can say that the length of a segment is the class of all segments that are equal to it. In any case, one should recognize the difference between equality and sameness when reading the *Elements,* where for example in Book 1, Proposition 35 is that parallelograms on the *same* base in the same parallels are equal, while Proposition 36 has the same conclusion when the parallelograms are on different but *equal* bases.

In the next simplest relation after equality, the greater magnitude is a *multiple* of the less. The less then *measures* the greater and is thus a *part* of the greater.

We can refine this relation. It is possible to say when some magnitude is the *same multiple* of another that a third magnitude is of a fourth. That "same multiple" could be the double, the triple, the quadruple, and so forth. The corresponding *same part* that the second magnitude is of the first and the fourth is of the third would be the half, the third, the quarter, and so on. This will be the basic case of *having the same ratio* or *being proportional.*

Suppose indeed A is the same multiple of B that C is of D. Despite my assertion that we must distinguish equality from sameness, we might also say that A and C are *equally multiple* of B and D respectively. Euclid's term here is usually ἰσάκις πολλαπλάσιος, the adverb ἰσάκις deriving from ἴσος "equal"; Heath translates Euclid's phrase as *equimultiple.* Euclid does use expressions that mean more literally "same multiple" in propositions 1 (τοσαυταπλάσιος) and 15 (ὡσαύτως πολλαπλάσιος) of Book v. For convenience, let us denote the situation of our magnitudes here by

$$A\,,\,B \text{ sm } C\,,\,D. \tag{3}$$

This is the basic case of what we we may write as either of

$$A : B :: C : D \tag{4}$$

and

$$B : A :: D : C,$$

saying A is to B as C is to D, or A has to B the (same) ratio that C has to D, or A, B, C, and D are proportional.

When two magnitudes are each a multiple of a third, they are *commensurable.* Suppose for example A and B are measured by E, and C and D are measured by F. If moreover

$$A\,,\,E \text{ sm } C\,,\,F, \qquad B\,,\,E \text{ sm } D\,,\,F, \tag{5}$$

then we still want to say that the proportion of (4) holds. We cannot do this, without having determined that something about A and B, some "bearing" of one to the other, is the *same* as it is about C and D. We have not really done this yet, because E and F are not uniquely determined by A and B and by C and D respectively. If E is the *greatest* common measure of A and B, and F is the same of C and D, in a situation that we may express by

$$\operatorname{gcm}(A, B) = E, \qquad \operatorname{gcm}(C, D) = F, \tag{6}$$

then (5) is equivalent to (4). This will be the meaning of propositions 5 and 6 of Book x.

In general, if a magnitude A is not measured by a less magnitude B of the same kind, still A exceeds some multiple of B by a remainder C that is less than B. Even in this case, A and B *have a ratio* in the sense of Book v, here with Heath's translation (the passage comes just after Euclid's initial description of ratio, quoted above):

> Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.
>
> Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

Implicitly, any two magnitudes of the same kind are assumed to have a ratio. This is so, even if the magnitudes are *incommensurable,* like the diagonal and side of a square.

Given a magnitude A and a less magnitude B of the same kind, we can always *try* to measure A by B. Or we can simply *measure* A by B, but the measurement may not be even, and then there will be a remainder C, which is less than B, as described above. In either case, the measurement of A by B is step one in the *Euclidean Algorithm.* If there is a remainder C, then step two is using C to measure B. If there is a remainder, D, then we proceed to step three, measuring C by D. If we can continue these steps *ad infinitum,* as contemplated in Proposition 2 of Book x, then A and B are incommensurable. Otherwise, by Proposition 3 of Book x, the last remainder found is the *greatest common measure* of A and B.

The same result is established for *numbers* in Proposition 2 of Book vii. Here a number is a *counting* number, a positive integer, except that Euclid's numbers are numbers of *things.* This is according to one of the definitions at the head of Book vii (here with the translation of Heath):

> Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.
>
> A number is a multitude composed of units.

"Number" here is *arithmos* as in "arithmetic," and "multitude" is *plēthos* as in "plethora" or indeed "full" (the Greek and English words are said to share an

Indo-European root). "Unit" is *monas, monad-* as in "monad." We could use that last word to translate Euclid's word, but John Dee coined the word "unit" for the purpose, as discussed elsewhere [33, §2.5, page 107].

The units here are all equal to some one magnitude fixed in advance. Euclid does not actually say that; however, it gives rigor to propositions 5–8 of Book x, whereby the commensurable magnitudes are precisely those that are in the ratio of numbers. According to the definition of numbers, it would seem, commensurable magnitudes would be instances of numbers, if their common measure were equal to the standard unit; however, it usually is not.

If we apply the Euclidean Algorithm to A and B, and then again to C and D, it may be that the steps are the same in either case. This means there is a remainder in the one case, whenever there is a remainder in the other, and furthermore, in all corresponding instances of measurement, the same multiples of the less magnitudes are taken from the greater.

We *could* say then that the proportions (4) hold. This would be the *anthyphaeretic* definition of proportion, ἀνθυφαίρεσις being the noun for what happens in the Euclidean Algorithm, namely *alternate subtraction.* Fowler's book [18] is all about this definition, though with little in particular about its application to numbers. The anthyphaeretic definition lies *behind* the formulation of sameness of ratio of commensurable magnitudes that I suggested above, whereby (4) means precisely (5), assuming (6). Such a formulation of sameness of ratio for *numbers* is the implicit meaning of the definition in Book VII of the *Elements*. The meaning is made more explicit in the demonstration of Proposition 4 of that book.

As for arbitrary magnitudes, the definition in Book V amounts to this, that (4) holds precisely when, for all equimultiples E and F of A and C that are at least as great as B and D respectively, it is equimultiples of B and D that, when taken from E and F, leave remainders less than B and D respectively.

To say more about this, let us note another definition from the head of Book VII (with my translation):

> Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.
>
> A number is said to multiply a number when, however many are the units in it, so many times be the [number] being multiplied set out, and some [number] be generated.

The product of A and B is thus $A \times B$, that is, A times B; alternatively, B, A times. Though Euclid does not do so explicitly, we can speak of the product of any magnitude by a number. For magnitudes then, (4) means that for every E that is a number or unit, there is some F, which is a number or unit or zero, such that

$$F \times B \leqslant E \times A < F \times B + B, \tag{7}$$

$$F \times D \leqslant E \times C < F \times D + D. \tag{8}$$

We have now extracted two definitions of (4) from the *Elements*:

General: (7) and (8) hold for some F that is a number or unit or zero, for every E that is a number or unit.

Numerical: (5) holds, where E and F are as in (6).

The numerical definition applies whenever A and B are commensurable, as are C and D.

The two definitions are equivalent whenever, in particular, A is a multiple of B, basically since every multiple of a multiple of B is a multiple of B:

$$E \times (G \times B) = (E \times G) \times B.$$

More generally then, if (4) holds numerically, so that (5) holds for some E and F, then

$$A : E :: C : F, \qquad E : B :: F : D,$$

both numerically and generally. In this case, (4) holds generally, *ex aequali* (δι' ἴσου "through the equal," as in Proposition v.22 for magnitudes – also, for numbers, Proposition vii.14, discussed in the next section).

So the numerical condition for proportion entails the general definition. The converse will hold, for numbers, once we know that multiplication of them is commutative, as in the next section.

4 Diagrams of numbers and notes

In the *Elements,* why does Euclid draw numbers as line segments? According to Reviel Netz in *The Shaping of Deduction in Greek Mathematics* [28, pages 42 and 43], the diagrams

> reflect a cultural assumption, that mathematics ought to be accompanied by diagrams. Probably line diagrams are not the best way to organise proportion theory and arithmetic. Certainly symbolic conventions such as '=', for instance, may be more useful. The lettered diagram functions here as an obstacle . . .

Netz himself may be reflecting a cultural assumption or prejudice. He has referred to Mueller in *Philosophy* *of Mathematics and Deductive Structure in Euclid's* Elements [27, p. 67]:

> One notes also that the diagram plays no real role in this proof [of Proposition vii.14, discussed below] except possibly as a mnemonic device for fixing the meaning of the letters. The nonfunctionality of the diagrams in arithmetic is an indication of the greater abstractness of the subject as compared with geometry. However, although in the present case the reasoning is purely logical, in many propositions combinatorial ideas and geometric analogies will be seen to be functioning.

I do not know how arithmetic is more abstract than geometry, when the first theorem we learn in life is an arithmetical one: all linear orderings of the same finite set are isomorphic to one another. We do not learn the theorem in that form, and we do not prove it, but it lies behind *counting* a pile of pebbles and always getting the same number. Euclid avoids having to prove this theorem by presenting numbers not as heaps of units, but as linear orders of them.

It is odd too that Mueller finds arithmetical diagrams "nonfunctional." Proposition vii.14 is that, if

$$A : B :: \mathit{\Delta} : E, \qquad B : \mathit{\Gamma} :: E : Z,$$

then by alternation (vii.13),

$$A : \mathit{\Delta} :: B : E, \qquad B : E :: \mathit{\Gamma} : Z,$$

and therefore

$$A : \mathit{\Delta} :: \mathit{\Gamma} : Z,$$

so by alternation again

$$A : \mathit{\Gamma} :: \mathit{\Delta} : Z.$$

The modern typography here is anachronistic. All the more so would modern fractional notation be. This seems to give us a simpler argument:

$$\frac{A}{B} = \frac{\mathit{\Delta}}{E}, \qquad \frac{B}{\mathit{\Gamma}} = \frac{E}{Z},$$

then

$$\frac{A}{B} \cdot \frac{B}{\mathit{\Gamma}} = \frac{\mathit{\Delta}}{E} \cdot \frac{E}{Z},$$

so "obviously,"

$$\frac{A}{\mathit{\Gamma}} = \frac{\mathit{\Delta}}{Z},$$

by "cancellation" – except cancellation is just what we are justifying in Proposition vii.14.

If modern notation lets us follow an argument in a visual or tactile way, so too do Euclid's diagrams, as in Figure 5. The diagram is copied from MS. D'Orville 301, the "Bodleian Euclid," which, according to the library website [12],

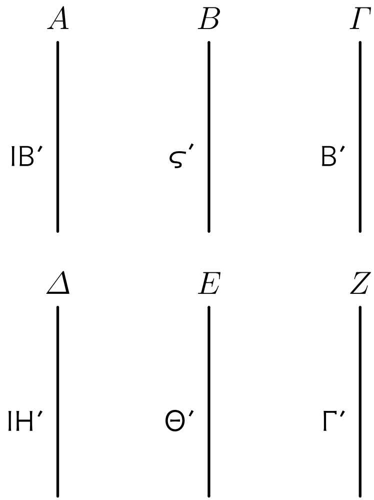

Figure 5: *Elements* VII.14

> was finished in September 888 by the scribe Stephanos Clericus, and was bought by Arethas of Patrae, Bishop of Caesarea. It is the oldest manuscript of a classical Greek author to bear a date, and the oldest surviving manuscript of the most widely studied version of Euclid – by Theon of Alexandria. It is a product of the great revival of scholarship in ninth-century Byzantium, in which many classical Greek texts were copied and thereby preserved for future generations. The text is beautifully written in a newly introduced cursive minuscule and accompanied by precise line drawings.

The equality of all six of the lines in the "precise drawing" in the original of Figure 5 could be due a copyist's mathematical ignorance; such at least is the general suggestion of Christián Carman, who experimented with how university students copied diagrams without knowing their meaning [9]. On the other hand, Ken Saito says of Euclid's diagrams for numbers [36, page 41],

> Their most salient feature is that the lines which stand for numbers are mostly of the same length, so that the lines in the diagram do not have the role of showing the metrical relations of the numbers they represent. What is significant is the arrangement of lines, which suggests the relationships between the numbers treated, and helps the reader's memory.

The lengths reproduced in Figure 5, which form the array

$$\begin{pmatrix} 12 & 6 & 2 \\ 18 & 9 & 3 \end{pmatrix},$$

were presumably added after Euclid, perhaps even by Bishop Arethas.

Proposition VII.1 of the *Elements* is that if the Euclidean Algorithm, which we described earlier, results in unity when applied to two numbers, then the numbers are prime to one another. After translating this, Heath remarks [14, Vol. 2, p. 297],

> the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

Heath is right that points should not be used, but if I understand him correctly, he is wrong about the reason. I think he means that Euclid's propositions are *general*, but that this would be belied by diagrams that indicated specific numbers. However, as they are, Euclid's proofs are often *not* general in form, regardless of the diagram. For example, in the proof of Proposition VII.1, there are *three* alternations of subtractions,

Figure 6: Multiplication of points

not some unspecified number of them. One must simply understand that there is nothing special about three alternations as being three. Throughout Book 1 of the *Elements,* specific triangles in the diagrams are to be understood as general. Specific numbers of dots in a diagram could be understood as general.

However, if units could always be thought of as points, then multiplication might obviously be commutative. Four rows of three dots each, or three rows of four dots each, as in Figure 6: it would seem evident that each array comprises the same number of dots.

Strictly what is evident is that the two sets of dots are in one-to-one correspondence with one another. As we have already noted, it is not obvious that every set has a unique *count,* meaning that all linear orderings of the set are isomorphic to one another. This is not obvious, in part because, as Cantor discovered [5, pages 103 & 104], it is not even *true* when the set is al-

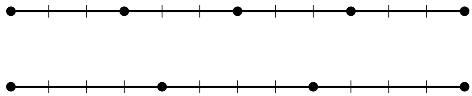

Figure 7: Products of line segments

lowed to be infinite. Euclid contemplates infinite sets, such as that of prime numbers, proved to be infinite in Proposition xi.20.

It is also not obvious that four line segments, of 3 units each, depicted in Figure 7, will have together the same length as three line segments, of 4 units each. The sameness is nonetheless a consequence of of Proposition vii.16.

Let us review the argument. If (3) holds, then so does

$$A\ ,\ B \text{ sm } A+C\ ,\ B+D,$$

by commutativity of addition: this is Proposition 5 of Book vii. Thus, if (5) holds, then

$$A\ ,\ E \text{ sm } A+C\ ,\ E+F,$$
$$B\ ,\ E \text{ sm } B+D\ ,\ E+F;$$

this is basically Proposition 6. Also, if (6) holds, then

$$\text{gcm}(A+C, B+D) = E+F,$$

though Euclid does not spell this out. In combination, propositions 5 and 6 yield that, if (4) holds for numbers, then so does

$$A : B :: A + C : B + D.$$

We can iterate this, obtaining what we may write as

$$A : B :: K \times A : K \times B, \tag{9}$$

though strictly, so formulated, this is Proposition 17. We are using

$$K \times A \,,\, A \text{ sm } K \times B \,,\, B.$$

As a special case of this, we have

$$A \,,\, 1 \text{ sm } A \times B \,,\, B, \tag{10}$$

and in this case, (9) becomes

$$1 : B :: A : A \times B.$$

Since 1 here measures B, so must A measure $A \times B$, the same number of times:

$$B \,,\, 1 \text{ sm } A \times B \,,\, A.$$

Meanwhile, we can interchange A and B in (10):

$$B \,,\, 1 \text{ sm } B \times A \,,\, A.$$

The last two results together give us

$$B \times A = A \times B. \tag{11}$$

Concerning such arguments in general, Barry Mazur reports [26, page 4],

> Often, in teaching Euclid's *Elements,* one emphasizes 'logical thinking' as the great thing that students take away from it. But I also see a type of pre-logical, if I can call it that, or intuition-enriching, benefit as well; this is hard to pinpoint but comes out as a sharpening of our ways of generally thinking about, guessing about, negotiating, comparing and relating, geometric objects.

Studying Euclid's proof of commutativity, one may develop the intuition of the one-to-one correspondence depicted in Figure 8.

We have shown in the previous section that the numerical condition for proportionality entails the general condition. For the converse, suppose (4) holds in the general sense, but A, B, C, and D are numbers. Then (7) holds when $E = B$ and $F = A$. Consequently, so does (8), and this means

$$A \times D = B \times C.$$

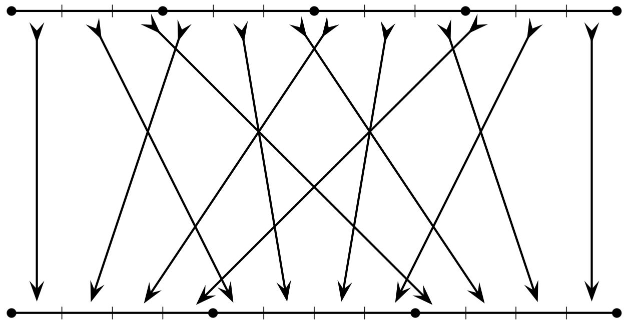

Figure 8: Commutativity by rearrangement

With this, along with (9) and (11), we have, in the numerical sense,

$$\begin{aligned} A : B &:: C \times A : C \times B \\ &:: A \times C : B \times C \\ &:: A \times C : A \times D \\ &:: C : D. \end{aligned}$$

Commutativity of multiplication of numbers gives us also, by the ordering of ratios in Book v, that (7) and (8) are equivalent to

$$\begin{gathered} F : E \leqslant A : B < (F + 1) : E, \\ F : E \leqslant C : D < (F + 1) : E. \end{gathered}$$

Thus we can understand ratios as positive real numbers, by the definition of these as *cuts* of the linear order of rational numbers. This is the definition that Dedekind works out without recourse to geometry [10].

5 Theoretical tuning

For the theory of proportion of numbers, we have seen how it is useful to consider first the ratios of pairs of numbers where the greater is a multiple of the less. Strictly, in Book vii of the *Elements,* Euclid considers *ordered* pairs of numbers, and these are of three different kinds, because a number can be, of another number,

1) a multiple (πολλαπλάσιος),
2) a part (μέρος), or
3) parts (μέρη).

Sameness of ratio is given thus (with my translation):

> Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.
>
> Numbers are proportional when the first of the second and the third of the fourth be equally multiple or the same part, or [there] be the same parts.

I am trying to reflect Euclid's use of *two* subjunctive verbs, singular and plural respectively, in the dependent clause. There does not seem to be any important *mathematical* distinction here, and Vitrac seems fine to use one verb, and this in the indicative mood [15, page 262]:

> Des nombres sont en proportion quand le premier, du deuxième, et le troisième, du quatrième, sont équimultiples, ou la même partie, ou les mêmes parties.

This is called a definition, but "same parts" is not defined; I think the meaning is brought out, as I have said, in Proposition 4 of Book VII, enunciated thus (with my translation):

> Ἅπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη.
>
> Every number, of every number, the less of the greater, is either part or parts.

The proof suggests the meaning, that the less number, when it does not measure the greater, consists of parts that are equal to the *greatest* common measure of the two numbers.

In the *Sectio Canonis,* concordant musical intervals have the ratio of numbers. There are also three kinds of such ratios, but they are not multiple, part, and parts. Rather, multiple and part are considered as one, and of the remaining ratios, there is a special kind where the less number goes *once* into the greater, leaving a remainder that measures the less.

When I was a child, and one thing was one-and-a-half times as much as another, I found it odd for my mother to say it was *half again* as much. Now I am using that expression as a pattern for translating the *Sectio Canonis*:

> πάντα δὲ τὰ ἐκ μορίων συγκείμενα **ἀριθμοῦ λόγῳ** λέγεται πρὸς ἄλληλα, ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν ἀριθμοῦ λόγῳ λέγεσθαι πρὸς ἀλλήλους· τῶν δὲ ἀριθμῶν οἱ μὲν ἐν **πολλαπλασίῳ λόγῳ** λέγονται, οἱ δὲ ἐν **ἐπιμορίῳ,** οἱ δὲ ἐν **ἐπιμερεῖ,** ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν τοῖς τοιούτοις λόγοις λέγεσθαι πρὸς ἀλλήλους.
>
> All things composed of parts are said to be **in numerical ratio** to one another; so notes too must be said to be in numerical ratio to one another. Some numbers are said to be in **multiple ratio,** others in **part-again,** still others in **parts-again** [ratio]; so notes too must be said to be in in these ratios to one another.

Where I have *part-again* and *parts-again* for ἐπιμόριος and ἐπιμερής, Barker uses the transliterations "epimoric" and "epimeric" [1, p. 192, p. 43 n. 63]. The words "superparticular" and "superpartient" might also be used, as for example in D'Ooge's translation of Nicomachus [29, pp. 52–3, 215, & 220]. The quoted passage is given in the big Liddell–Scott–Jones lexicon [24] as an example of the use of ἐπιμόριος. This word is derived from μόριον rather than μέρος; but either of these means "part."

Numbers A and B, the latter the less, are in

1) *multiple* ratio, if A is a multiple of B;
2) *part-again* ratio, if $A - B$ measures B, so that A is the whole of B with a part added;
3) *parts-again* ratio, otherwise.

Euclid continues somewhat obscurely.

> τούτων δὲ οἱ μὲν πολλαπλάσιοι καὶ ἐπιμόριοι ἑνὶ ὀνόματι λέγονται πρὸς ἀλλήλους.
>
> Of these, the multiple and the part-again are said with a single name to one another.

Barker's translation is, "of these, the multiple and the epimoric are spoken of in relation to one another under a single name"; Fowler changes "in relation" to "with respect" [18, page 139]. Barker suggests that "single name" refers to the single words like τριπλάσιος ("triple") and ἐπίτριτος ("third-again") that name specific multiple and part-again ratios. Fowler goes further, suggesting that "single name" alludes to the single number that specifies one of these ratios, as the number three specifies the triple and third-again ratios.

Fowler's suggestion would seem to be partially corroborated by the last three propositions of Book VII of the *Elements.* The last of these, Proposition 39, is a *problem,* in the sense of something to be done. It starts out as follows (interwoven with my translation).

> Ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ δοθέντα μέρη.
>
> To find the number that, being least, will have given parts.
>
> Ἔστω τὰ δοθέντα μέρη τὰ Α, Β, Γ·
>
> [Let] be the given parts A, B, and Γ.
>
> δεῖ δὴ ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ Α, Β, Γ μέρη.
>
> One must then find the number that, being least, will have parts A, B, and Γ.
>
> Ἔστωσαν γὰρ τοῖς Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ οἱ Δ, Ε, Ζ, καὶ εἰλήφθω ὑπὸ τῶν Δ, Ε, Ζ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η.
>
> [Let] be indeed, to the parts A, B, and Γ, the homonymous numbers Δ, E, and Z, and [let] have been taken the least number measured by Δ, E, and Z, [namely] H.

Least common multiples of pairs and triples of numbers were constructed in propositions 34 and 36. Since we also know multiplication is commutative,

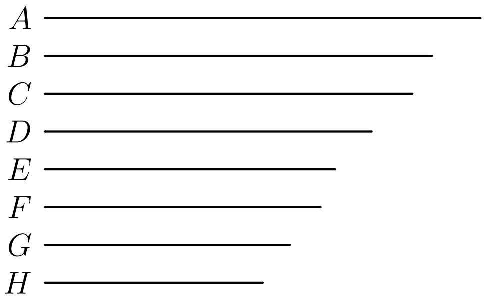

Figure 9: Lyre strings as numbers

Proposition 39 gives us nothing new. Moreover, its diagram features a line segment for each of the eight letters named. It would seem that A represents the ratio $1 : \mathit{\Delta}$; B, $1 : B$; and $\mathit{\Gamma}$, $1 : \mathit{\Gamma}$. Thus either Euclid is not as sensitive to the abstractness of ratios as I thought, or somebody else added Proposition 39 later to Book VII.

As for music, it seems the traditional lyre had seven strings, but some could have eight or even more [2, pp. 15 & 276]. Suppose eight strings, spanning a diapason or octave, are represented by numbers A through H, as in Figure 9. As we have noted from [2, page 13], A, D, E, and H are fixed, while the other strings can vary according to the chosen musical mode. The fourths AD and EH, the fifths AE and DH, and the octave or diapason AH are all *concords,* and the octave is composed of a fourth and a fifth. The double octave is also a concord, but the double fourth and the double fifth are not. So much is observed or at least assumed.

Following Euclid, or whoever wrote the *Sectio Canonis*, we propose the axiom that a concord corresponds to a ratio of numbers that is either multiple or part-again.

The least numbers that are in the ratio of numbers A and B must measure A and B respectively: this is Proposition 20 of Book VII of the *Elements.* Indeed, suppose

$$A : B :: C : D,$$

and C and D are least in that capacity, but C is *parts* of A. Then each of the parts is equal to the greatest common measure of C and A. This being E, as that of D and B is F, we have

$$C : D :: E : F,$$

so that

$$A : B :: E : F$$

– but in that case, C and D were not least.

The foregoing argument has been questioned [31, 25], because of its reliance on transitivity of sameness of ratio, which is not obvious if (4) is defined for numbers when (5) holds for *some* E and F. This is why I say that the condition (6) must also be understood. Euclid would not define two things (such as ratios or being "parts") to be the same unless the sameness so defined were obviously transitive. One reason to say this is the care with which he distinguishes equality and sameness.

Proposition VII.20 leads to what is established by the opening propositions of Book VIII, that every finite geometric sequence of numbers has, in modern notation, the form of a row of the array in Figure 10. Here a and b are prime to one another, and c is arbitrary. In particular, if (4) holds for numbers, and A and B are the first and last terms of a geometric sequence of given length, then so are C and D: this is Proposition VIII.8.

A part-again ratio is a ratio $A : B$, where A is a unit more than B, so that no number can fit between these two. In particular, no geometric sequence with three (or more) terms can start with A and end with B. The double octave corresponds to a geometric sequence of three terms. Therefore, by what we have just seen, notes that are a double octave apart cannot be in part-

$$c$$

$$ac \qquad bc$$

$$a^2c \qquad abc \qquad b^2c$$

$$a^3c \qquad a^2bc \qquad ab^2c \qquad b^3c$$

$$a^4c \qquad a^3bc \qquad a^2b^2c \qquad ab^3c \qquad b^4c$$

Figure 10: Geometric sequences

again ratio. By our axiom then, they can only be in multiple ratio. The same then is true of notes that are an octave apart.

By that argument in contrapositive form, the fifth and the fourth cannot be multiple, since their doubles are not. Therefore the fifth and the fourth are part-again. Together, they make up an octave. The least multiple ratio, namely the double, is composed of the two greatest part-again ratios, namely the half-again (ἡμιόλιος, half-whole or "hemiolic") and the third-again (ἐπίτριτος, "epitritic"). Therefore these three ratios must correspond respectively to the octave, the fifth, and the fourth.

6 Platonic scale

In the *Timaeus* dialogue (35b–6b), Plato has the title character work out in another way the musical ratios that we have just found [34, 35]. Briefly, from the low end to the high, the octave is divided into a fourth and fifth by the *arithmetic* mean of the length of the strings playing it; fifth and fourth by the *harmonic* mean. We can fill out the octave with four more notes to get a *diatonic* scale, then five more for a *chromatic* scale, by taking, in an interval $[a, 2a]$, the residues, with respect to the *multiplicative* modulus 2, of consecutive integer powers of 3.

The speech of Timaeus concerns activities of God or “the god” (ὁ θεός), as follows.

ἤρχετο δὲ διαιρεῖν ὧδε.	He began dividing thus:
μίαν ἀφεῖλεν τὸ πρῶτον	*One* he took first
ἀπὸ παντὸς μοῖραν,	from the whole, [one] part;
μετὰ δὲ ταύτην	after this,
ἀφῄρει διπλασίαν ταύτης,	he took double of this;
τὴν δ᾽ αὖ τρίτην	third,
ἡμιολίαν μὲν τῆς δευτέρας,	half again of the second,

τριπλασίαν δὲ τῆς πρώτης,	triple of the first;
τετάρτην δὲ	fourth,
τῆς δευτέρας διπλῆν,	of the second, double;
πέμπτην δὲ	fifth,
τριπλῆν τῆς τρίτης,	triple of the third;
τὴν δ᾽ ἕκτην	sixth,
τῆς πρώτης ὀκταπλασίαν,	of the first, eightfold;
ἑβδόμην δ᾽	seventh,
ἑπτακαιεικοσιπλασίαν	seven-and-twentyfold
τῆς πρώτης·	of the first.

If the original part is A, then all of the parts taken are respectively

$$A, \quad 2A, \quad 3A, \quad 4A, \quad 9A, \quad 8A, \quad 27A.$$

The last is the sum of the others – Archer-Hind notes this, along with supposedly Pythagorean interpretations of the individual numbers that will not concern us [34, page 107]. We just analyze the list into two geometric sequences sharing the initial term, as in Figure 11. Meanwhile, Timaeus continues:

μετὰ δὲ ταῦτα	After this
συνεπληροῦτο	he filled out

τά τε διπλάσια	the double
καὶ τριπλάσια διαστήματα,	and triple intervals,
μοίρας ἔτι ἐκεῖθεν	more parts from there
ἀποτέμνων	cutting off
καὶ τιθεὶς	and putting them
εἰς τὸ μεταξὺ τούτων,	in the middle of these,
ὥστε ἐν ἑκάστῳ διαστήματι	that in each interval
δύο εἶναι μεσότητας,	there be two means:
τὴν μὲν	the one,
ταὐτῷ μέρει	by the same part
τῶν ἄκρων αὐτῶν	of the extremes themselves
ὑπερέχουσαν	exceeding
καὶ ὑπερεχομένην,	and exceeded,
τὴν δὲ	the other,
ἴσῳ μὲν κατ᾽ ἀριθμὸν	equally numerically
ὑπερέχουσαν,	exceeding,
ἴσῳ δὲ ὑπερεχομένην.	equally exceeded.

The idea seems to be as follows. Suppose we have a length OA, divided randomly at B, as in Figure 12. We divide BA at two points, H and M, so that

- the excess of OH over OB is to the latter as the excess of OA over HA is to the former;
- the excess of OM over OB is equal to the excess of OA over OM.

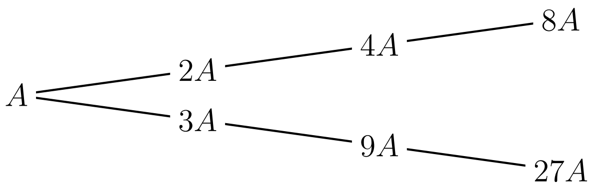

Figure 11: Two geometric sequences

Thus

$$BH : OB :: HA : OA, \tag{12}$$
$$BM = MA. \tag{13}$$

Then OH and OM are called respectively the *harmonic* and the *arithmetic mean* of OB and OA. By manipulations justified by *Elements* v.16 and 18, we have the equivalence of (12) and

$$\begin{gathered} OH : OB :: HA + OA : OA, \\ OH : HA + OA :: OB : OA, \\ OH : 2OA :: OB : OB + OA, \\ OH : OB :: 2OA : OB + OA. \end{gathered}$$

Thanks to (13), the last proportion is equivalent to

$$OH : OB :: OA : OM.$$

The *geometric* mean of OB and OA is their mean proportional, OG, so that

$$OB : OG :: OG : OA.$$

By VI.16, the last two proportions are equivalent to

$$OH \cdot OM = OB \cdot OA = OG^2. \qquad (14)$$

Thus the geometric mean of two magnitudes is the geometric mean of their other two means:

$$OH : OG :: OG : OM.$$

We can construct all three means as in Figure 12, where again we are given OA, divided arbitrarily at B. The midpoint of AB is M. The perpendicular at B meets the circle whose diameter is OA at C, and then angle OCA is right by *Elements* III.31. In this case,

$$OC^2 = OB \cdot OA$$

by VI.8. Thus the circle through C that has center O meets OA at G. The latter circle meeting at D the circle whose diameter is BA, we have

$$OD^2 = OB \cdot OA.$$

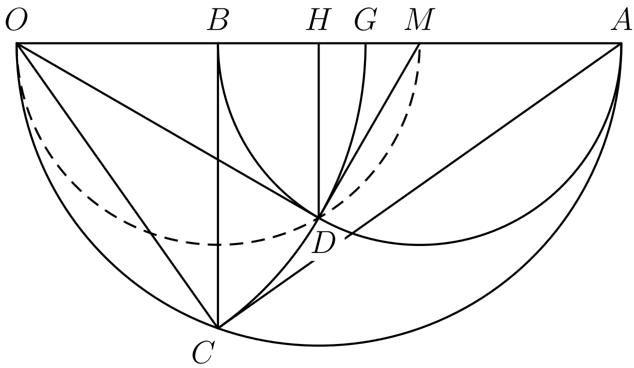

Figure 12: Three means

Thus the perpendicular dropped from D meets OA at H. Indeed, by *Elements* III.37, OD is tangent to the circle whose diameter is BA. That circle having center M, the angle ODM is right and therefore

$$OD^2 = OH \cdot OM.$$

Since $OD = OG$, we have (14). Also the circle with diameter OM passes through D.

To work out an alternative construction, as will be given in Figure 13, we suppose first

$$OA \cdot OA' = OB \cdot OB' = OH \cdot OH'.$$

Then

$$\begin{aligned}
OA \cdot (OA' + OB') &= OA \cdot OA' + OA \cdot OB' \\
&= OB \cdot OB' + OA \cdot OB' \\
&= (OB + OA) \cdot OB' \\
&= 2OM \cdot OB',
\end{aligned}$$

so

$$\begin{aligned}
OA' + OB' : OB' &:: 2OM : OA \\
&:: 2OB : OH,
\end{aligned}$$

and finally

$$\begin{aligned}
OH \cdot (OA' + OB') &= 2OB \cdot OB' \\
&= 2OH \cdot OH',
\end{aligned}$$

so that

$$2OH' = OA' + OB'.$$

In modern terms, the inverse of the harmonic mean of two magnitudes is the arithmetic mean of the inverses of the magnitudes. Thus we can construct the harmonic mean of OA and OB as in Figure 13, where, given OA divided at B, we have drawn OB' at an angle to OA, then found A' on OB', by means of *Elements* 1.43, so that parallelograms $A'A$ and $B'B$ are

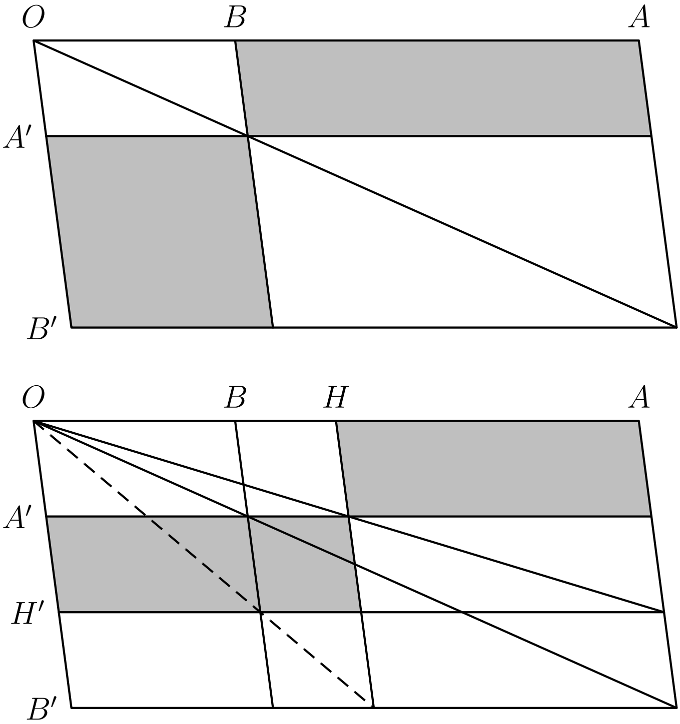

Figure 13: Harmonic mean

equal. When H' is the midpoint of $A'B'$, we obtain H as in the diagram by completing the parallelogram $H'A$, drawing the diagonal from O, seeing where it cuts the base of parallelogram $A'A$, and letting the part on the side of A' be the base of parallelogram $A'H$, so that parallelograms $H'H$ and $A'A$ are equal (as then are $H'H$ and $B'B$).

If, on the canon or monochord, we want to divide an interval of ratio $a : b$ at the harmonic and arithmetic means, where a and b are numbers prime to one another, we can mark out whole-number distances as

$$2a(a+b), \qquad 4ab, \qquad (a+b)^2, \qquad 2b(a+b).$$

if $a + b$ is even, we can divide by 4. According to Timaeus, the god divides double and triple intervals at their harmonic and arithmetic means. The octave then is marked out at relative distances

$$6, \qquad 8, \qquad 9, \qquad 12,$$

so the intervals are a fourth, a tone, and a fourth, respectively. The triple interval, or twelfth, can be marked at

$$2, \qquad 3, \qquad 4, \qquad 6,$$

the intervals being fifth, fourth, and fifth. Now the god further divides each fourth, as we have already discussed, marking off two tones, of ratio $8 : 9$, and thus leaving a leimma, of ratio $243 : 256$, which is 3^5 to 2^8:

ἡμιολίων δὲ διαστάσεων	Half-again intervals,
καὶ ἐπιτρίτων	and third-again,
καὶ ἐπογδόων	and eighth-again
γενομένων	coming to be
ἐκ τούτων τῶν δεσμῶν	from these bonds
ἐν ταῖς πρόσθεν διαστάσεσιν,	in the original intervals,
τῷ τοῦ ἐπογδόου	with the eighth-again
διαστήματι	interval
τὰ ἐπίτριτα πάντα	all the third-again
συνεπληροῦτο,	he filled out,
λείπων αὐτῶν ἑκάστου	**leaving** from them each
μόριον,	a part –
τῆς τοῦ μορίου	of that part
ταύτης διαστάσεως	the same interval
λειφθείσης	**being left**
ἀριθμοῦ	that, of number
πρὸς ἀριθμὸν ἐχούσης	for number, has
τοὺς ὅρους	the terms
ἓξ καὶ πεντήκοντα	6 + 50
καὶ διακοσίων πρὸς	+ 200 to
τρία καὶ τετταράκοντα	3 + 40

καὶ διακόσια.	+ 200.

Barker suggests [1, page 38, note 36] that Plato calls that last interval a *leimma,* apparently because he uses participles, which I have bolded, from the verb cognate verb λείπω "leave." (We see this verb in "ellipsis," a transliteration of ἐλλείψις.)

One may observe

$$27 : 1 :: (9 : 8)(3 : 2)(2 : 1)^4,$$

so that Plato has the god of Timaeus fill out four octaves and a major sixth. Because the sequences of powers of 2 and 3 end with the cubes, one may speculate that the three dimensions of the material world are being alluded to [34, page 111].

7 Powers of three

We can also obtain all of the notes that the god does, and more, without taking harmonic or arithmetic means as such, but just by continuing the sequence of powers of three, then scaling by powers of two.

We can start doing this as in Table 5, where col-

Table 5: Diatonic scale

1	2	3	4	5	6	7	8	9	10	11
2	4	8	16	32	64	128	256	512	1024	2^{11}
							243	486	972	$2^3 3^5$
				27	54	108	216	432	864	$2^6 3^3$
	3	6	12	24	48	96	192	384	768	$2^9 3$
									729	$3^6 2$
						81	162	324	648	$2^4 3^4$
			9	18	36	72	144	288	576	$2^7 3^2$
1	2	4	8	16	32	64	128	256	512	2^{10}

umn n features the residues, with respect to the *multiplicative* modulus 2, of powers of 3 in the interval $[2^{n-1}, 2^n]$. We have notes, in

column 1, delineating the octave;

columns 2 and 3, dividing the octave into fifth and fourth;

column 4, dividing the octave into a tone and two fourths, as in Figure 2, and constituting what Carey and Clampitt [6] call a *structural* scale;

column 7, belonging to a *pentatonic* scale, dividing the octave into three tones and two minor thirds;

columns 10 and 11, of a *heptatonic* or *diatonic* scale, dividing the octave into fives tones and two leimmata.

We have understood a tone to be an interval with ratio $9 : 8$; however, with terms such as *pentatonic,* a tone is apparently a note. In any case, each of our scales here is a division of the octave, by multiplicative residues of powers of three, into intervals of just two sizes. In the other columns, there are three different sizes. That there can be no more than three sizes is a general result, established in the 1950s and reviewed in the next section.

If the numbers in the columns of Table 5 are understood as frequencies, then the seven notes of the diatonic scale, from lowest on up (that is, from the bottom of the table), could be those called today

F, G, A, B, C, D, E,

with F then reappearing at the top. If we rise through that scale by thirds, these are major and minor in the pattern of Table 1 and Figure 1:

F 1 A 0 C 1 E 0 G 1 B 0 D 0 F

The ratios of the thirds are $3^4 : 2^6$ and $2^5 : 3^3$. If one wanted to make them concordant, according to the axioms of the *Sectio Canonis,* one might require

Table 6: Chromatic scale

12	13	14	15	16	17	18	19
2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}
$2^4 3^5$	$2^5 3^5$	$2^6 3^5$	$2^7 3^5$	$2^8 3^5$	$2^9 3^5$	$2^{10} 3^5$	$2^{11} 3^5$
				$\mathbf{3^{10}}$	$3^{10} 2$	$2^2 3^{10}$	$2^2 3^{11}$
$2^7 3^3$	$2^8 3^3$	$2^9 3^3$	$2^{10} 3^3$	$2^{11} 3^3$	$2^{12} 3^3$	$2^{13} 3^3$	$2^{14} 3^3$
	$\mathbf{3^8}$	$3^8 2$	$2^2 3^8$	$2^3 3^8$	$2^4 3^8$	$2^5 3^8$	$2^6 3^8$
$2^{10} 3$	$2^{11} 3$	$2^{12} 3$	$2^{13} 3$	$2^{14} 3$	$2^{15} 3$	$2^{16} 3$	$2^{17} 3$
$2^2 3^6$	$2^3 3^6$	$2^4 3^6$	$2^5 3^6$	$2^6 3^6$	$2^7 3^6$	$2^8 3^6$	$2^9 3^6$
						$\mathbf{3^{11}}$	$3^{11} 2$
$2^5 3^4$	$2^6 3^4$	$2^7 3^4$	$2^8 3^4$	$2^9 3^4$	$2^{10} 3^4$	$2^{11} 3^4$	$2^{12} 3^4$
			$\mathbf{3^9}$	$3^9 2$	$2^2 3^9$	$2^3 3^9$	$2^4 3^9$
$2^8 3^2$	$2^9 3^2$	$2^{10} 3^2$	$2^{11} 3^2$	$2^{12} 3^2$	$2^{13} 3^2$	$2^{14} 3^2$	$2^{15} 3^2$
$\mathbf{3^7}$	$3^7 2$	$2^2 3^7$	$2^3 3^7$	$2^4 3^7$	$2^5 3^7$	$2^6 3^7$	$2^7 3^7$
2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}

these ratios to be $6 : 5$ and $5 : 4$. Nonetheless, we are working only with factors of 2 and 3.

If we keep going, as in Table 6, we end up (in columns 18 and 19) dividing each of the five tones of the diatonic scale. In particular, under the frequency interpretation, for the higher note of each tone, we introduce a *flattened* variant, lower by a leimma. Thus we obtain a twelve-tone or *chromatic* scale, whose notes could be

$$\text{F, G}\flat\text{, G, A}\flat\text{, A, B}\flat\text{, B, C, D}\flat\text{, D, E}\flat\text{, E.}$$

These divide the octave into seven leimmata and five instances of the *apotomē* – what we "cut off" from a tone when we leave behind a leimma.

Alternatively, understanding each note as the *sharpened* variant of its own flattened variant, we can interpret the same notes as

$$\text{F}\sharp\text{, G, G}\sharp\text{, A, A}\sharp\text{, B, C, C}\sharp\text{, D, D}\sharp\text{, E, F.}$$

We may arrange these in a ring, as in Figure 14, where each solid arrow sends us either up a fifth or down a fourth, and the dashed arrow sends us down a little less than a fourth, by (1) or (2).

Today, the 12-note scale is (usually) *equally tempered,* so that all twelve intervals that make up the octave are equal. In that case, the numbers around the circle in Figure 14 become consecutive powers of $\sqrt[12]{2}$, and we need only consider their exponents *modulo* 12. The arrows in the diagram then exhibit scaling of those exponents by 7, and this scaling is an automorphism of the cyclic group of integers *modulo* 12. There are similar results for the structural, pentatonic, and diatonic scales, but not for other numbers of fewer than twelve notes obtained as residues

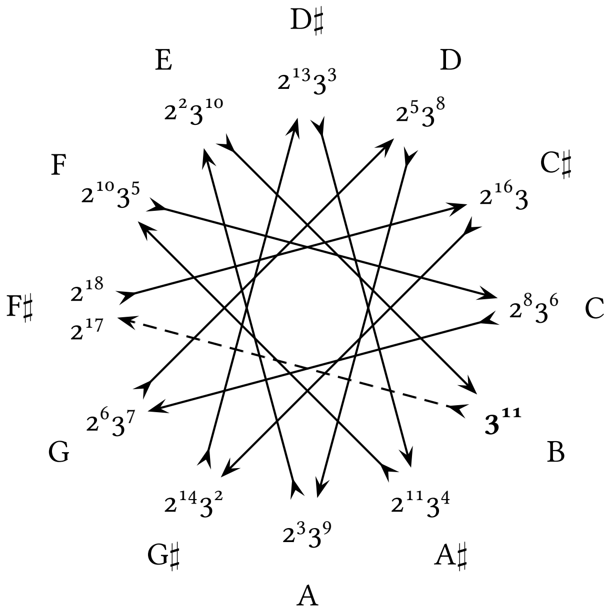

Figure 14: Circle of fifths

of consecutive powers of 2. Carey and and Clampitt [6, 7, 8] give a characterization of this situation that we shall also review in the next section. The condition will be equivalent to the one that we have already suggested: having intervals of only two sizes, not three.

Meanwhile, *not* tempering our notes, we get more of them in new columns after those of Table 6. We end up not with a *circle* of fifths, but a *spiral* [4, § 5.3, page 156]. First we have 3^{12}, which

- falls short of $2^8 3^7$ by a leimma;
- is in excess of 2^{19} by the same interval by which
  - six tones are in excess of an octave, by (2);
  - a tone is in excess of two leimmata.

The ratio $3^{12} : 2^{19}$ is that of the so-called *Pythagorean comma.* Continuing, we divide each apotome into leimma and comma. We get a 17-note scale, in which the octave is made up of five Pythagorean commas and 12 leimmata. The scale is shown in Figure 15. Here we have wrapped our number-line into a spiral, sending each number n to the point

$$\left(n^a; \frac{2\pi \log n}{\log 2}\right)$$

in polar coordinates. The parameter a, which sets the density of the arms of the spiral. Numbers that are oc-

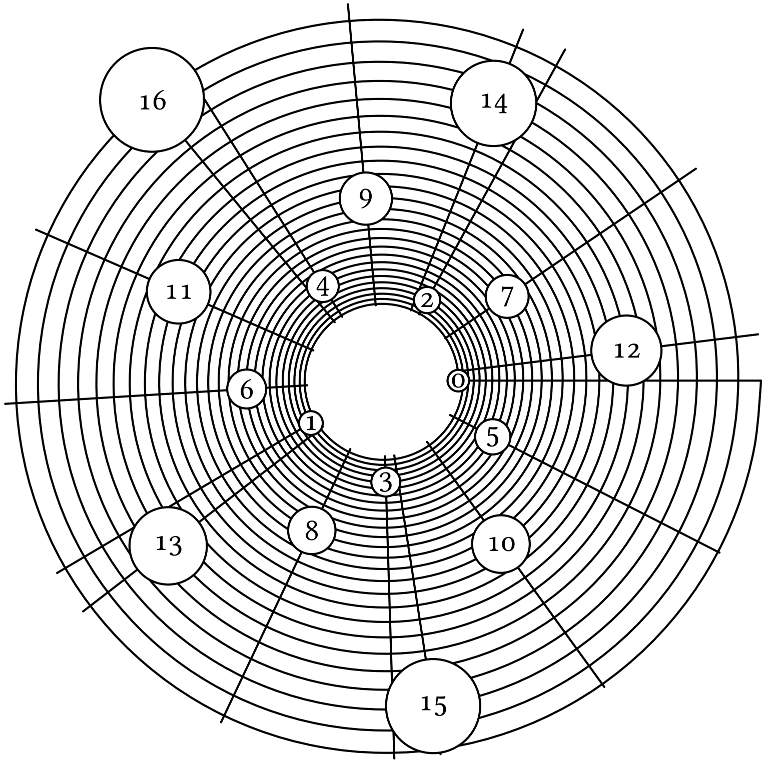

Figure 15: Spiral of sixteen fifths

taves apart – numbers whose ratio is a power of 2 – are precisely those that end up on the same ray originating from the origin. In an alternative formulation, the spiral contains the points

$$\left(r, \frac{2\pi \log r}{a \log 2}\right).$$

The polar equation of the spiral itself is

$$r = b^{\vartheta}, \tag{15}$$

where

$$b = 2^{a/2\pi}.$$

In our actual figure, we have used for b the value $5^{1/52\pi}$, so that 26 turns around the spiral set us five times further away from the origin than we started. When $0 \leqslant n < 17$, we have indicated in the figure the point where 3^n lands; its angle is given by

$$\vartheta = \frac{2n\pi \log 3}{\log 2}.$$

Each of the five notes that succeeded a sharp note in the last interpretation has now been given a flat variant too, still exceeding that sharp note by a Pythagorean comma. This makes all of the notes

Table 7: Flats and sharps

$2^{16}3^{6}$	C
$2^{8}3^{11}$	B
$\mathbf{3^{16}}$	B♭
$2^{19}3^{4}$	A♯
$2^{11}3^{9}$	A
$2^{3}3^{14}$	A♭
$2^{22}3^{2}$	G♯
$2^{14}3^{7}$	G
$2^{6}3^{12}$	G♭
2^{25}	F♯
$2^{17}3^{5}$	F
$2^{9}3^{10}$	E
$2^{1}3^{15}$	E♭
$2^{20}3^{3}$	D♯
$2^{12}3^{8}$	D
$2^{4}3^{13}$	D♭
$2^{23}3^{1}$	C♯
$2^{15}3^{6}$	C

as in Table 7. Since the leimma is so much bigger than the comma, an equally tempered seventeen-note scale might not make much practical sense. Its "fifth" would have the ratio $2^{10/17}$, because

$$3^{17} = 129\,140\,163, \qquad 2^{17+10} = 134\,217\,728.$$

The "circle of fifths" would be a projection of the spiral in Figure 16. According to angle in the polar coordinate system, the order of fifths remains the same:

0, 12, 7, 2, 14, 9, 4, 16, 11, 6, 1, 13, 8, 3, 15, 10, 5.

These are residues *modulo* 17 of successive multiples of 12, this being the inverse of 10 with respect to that modulus. Since 12 has no inverse *modulo* 16, there is no equally tempered 16-note version of the scale based on the first fifteen powers of three.

If we keep going after the sixteenth power, we cut Pythagorean commas out of the leimmata. We have so far relied on the following computations, which are noted by Carey and Clampitt [6, Table 3, page 119] and by Benson [4, § 5.2, page 155]. The excess of

- a triple interval over an octave is a fifth;
- an octave over a fifth is a fourth;
- a fifth over a fourth is a tone;
- a fourth over two tones is a leimma;

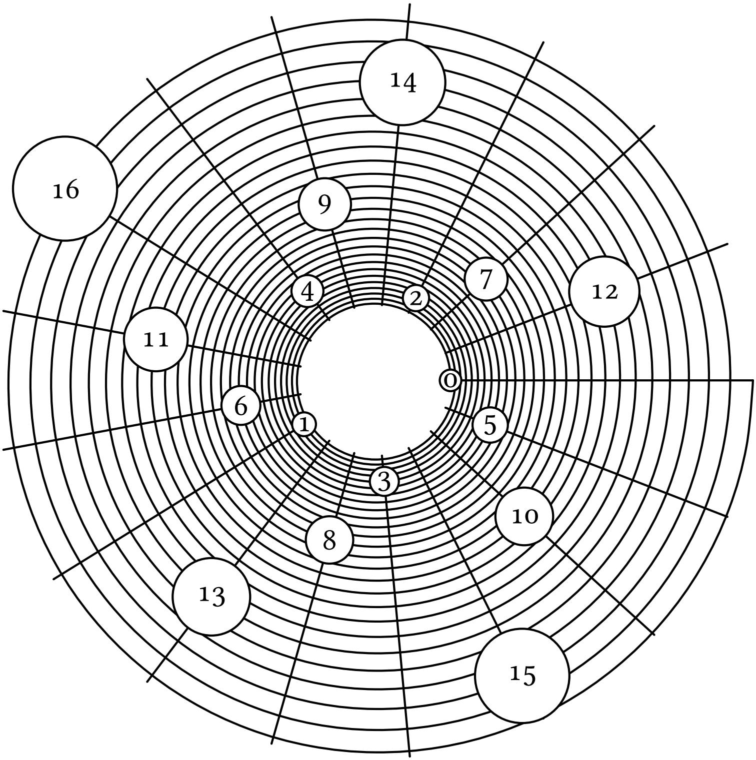

Figure 16: Circle of 17 equally tempered fifths

- a tone over two leimmata is a Pythagorean comma.

These are the steps of the Euclidean Algorithm, from books VII and X of the *Elements,* applied to the triple inverval and the octave. Numerically, multiplicatively,

$$3 = 2 \cdot \frac{3}{2}, \qquad 2 = \frac{3}{2} \cdot \frac{4}{3}, \qquad \frac{3}{2} = \frac{4}{3} \cdot \frac{9}{8},$$

and then

$$\frac{4}{3} = \left(\frac{9}{8}\right)^2 \cdot \frac{2^8}{3^5}, \qquad \frac{9}{8} = \left(\frac{2^8}{3^5}\right)^2 \cdot \frac{3^{12}}{2^{19}}.$$

Additively, in the usual form for the Algorithm, with one more step, we have the equations of Table 8. Thus, by taking 36 more powers of 3, as in Figure 17 (where we now dispense with drawing the spiral line through the powers), we take three Pythagorean commas out of each leimma. The result is a 53-note scale, dividing the octave into 41 Pythagorean commas and 12 smaller intervals, shaded in the figure, each having ratio $2^{65} : 3^{41}$.

We could have obtained scales of 41 and 29 notes by removing two commas or just one, respectively, from each leimma. This would amount to erasing 12 and then 12 more of the radial lines in Figure 17. The first

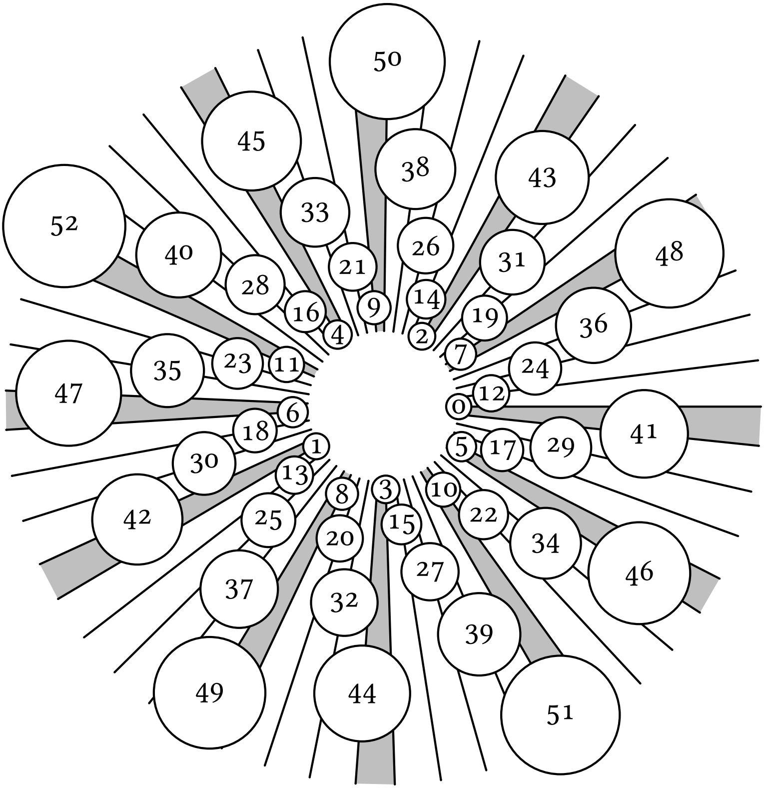

Figure 17: Scale of 53 notes

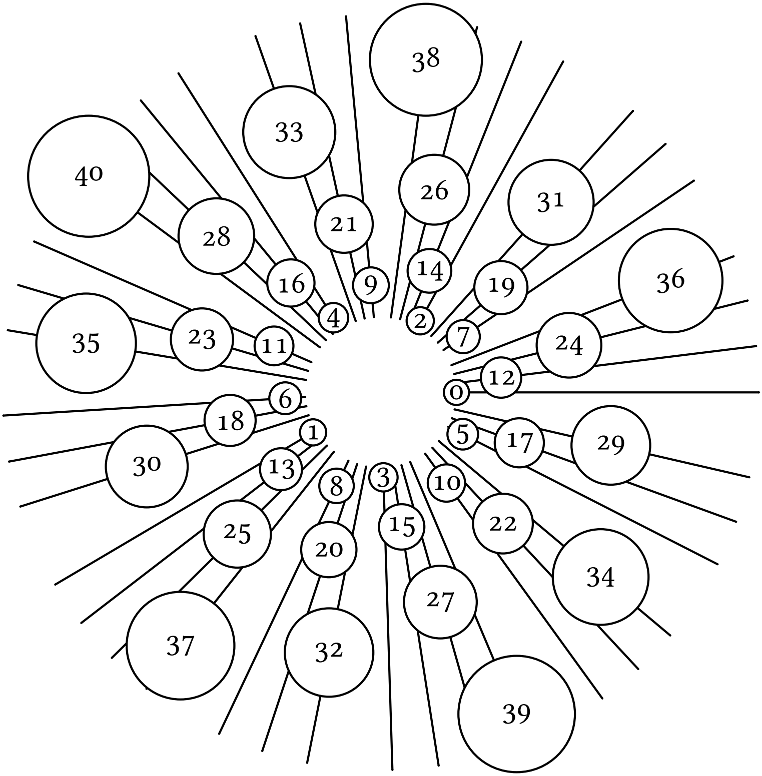

Figure 18: Scale of 41 notes

Table 8: Euclidean Algorithm for $\log_2 3$

$$\begin{aligned}
\log(3/1) &= \log(2/1) + \log(3/2) \\
\log(2/1) &= \log(3/2) + \log(2^2/3) \\
\log(3/2) &= \log(2^2/3) + \log(3^2/2^3) \\
\log(2^2/3) &= 2\log(3^2/2^3) + \log(2^8/3^5) \\
\log(3^2/2^3) &= 2\log(2^8/3^5) + \log(3^{12}/2^{19}) \\
\log(2^8/3^5) &= 3\log(3^{12}/2^{19}) + \log(2^{65}/3^{41})
\end{aligned}$$

result is as in Figure 18, where the two different sizes of intervals should be obvious. Dunne and McConnell write about this scale [11]:

> experiments with "microtonality" in twentieth-century music have tended towards dyadic parts of the half-step (scales with twenty-four tones, forty-eight tones, etc.). The most popular invention seems to be the quarter-tone. In other words, we are still tied to 12. The failure to employ the more natural microtones of the 41-tone scale is clear and compelling evidence of the continuing need for mathematics across the curriculum, especially the need for Number Theory in Music.

The scale of 53 notes would seem to be more natural still. Benson devotes a section to it [4, § 6.3, pages

213–7]. Strictly, the devotion is to the equally tempered scale, but this is quite close to the Pythagorean one of Figure 17. We can see there the other numbers of notes that we have considered, particularly

5, in the pentagon whose vertices, numbered 48, 50, 52, 49, and 51, are at the ends of spirals of the other notes;

7, in the heptagon we get with the additional vertices 47 and 46, ending spirals going the other way;

12, in the 12 shaded strips and corresponding steep spirals.

Different parameters b in the polar equation (15) may change the relative prominence of the families of spirals of notes.

8 Continued fractions

We repeat now in general terms what we did in the previous section. We are somehow still doing ancient mathematics, while relying on modern notation. Much of it is standard, and then I am going to be as succinct as I can, while not assuming anything that has not already been assumed. Matrix notation here may not be standard, but is useful, perhaps even as the diagram that we considered in Figure 5 is useful.

We can understand the real numbers $\log 3$ and $\log 2$ as instances of incommensurable magnitudes ξ_0 and ξ_1, the latter the less. By the Euclidean Algorithm, the magnitudes become the first terms of an infinite decreasing sequence $(\xi_0, \xi_1, \dots)$. At the same time, an infinite sequence $(a_0, a_1, \dots)$ of positive integers is produced: this is the **anthyphaeretic sequence** of the magnitudes. The two sequences respect the rule

$$\xi_k = \xi_{k+1} \cdot a_k + \xi_{k+2}. \tag{16}$$

We have seen in Table 8 that the anthyphaeretic sequence of $\log 3$ and $\log 2$ begins with $(1, 1, 1, 2, 2, 3)$.

It will be convenient write (16) in two more ways, first with matrices, thus:

$$\xi_k = \begin{pmatrix} a_k & 1 \end{pmatrix} \begin{pmatrix} \xi_{k+1} \\ \xi_{k+2} \end{pmatrix}. \tag{17}$$

This gives us

$$\begin{pmatrix} \xi_k \\ \xi_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_{k+1} \\ \xi_{k+2} \end{pmatrix}. \tag{18}$$

Next, for **continued fractions,** we introduce the notation

$$[a] = a, \qquad [a, b, \dots, z] = a + \frac{1}{[b, \dots, z]}, \tag{19}$$

where $a, \ldots, z$ are real numbers, and all but a are required to be positive. From (16) or (18) we derive

$$\frac{\xi_k}{\xi_{k+1}} = \left[a_k, \frac{\xi_{k+1}}{\xi_{k+2}}\right]. \tag{20}$$

This yields

$$a_k < \frac{\xi_k}{\xi_{k+1}}, \tag{21}$$

as well as, by repeated application,

$$\frac{\xi_0}{\xi_1} = \left[a_0, \ldots, a_{k-1}, \frac{\xi_k}{\xi_{k+1}}\right]. \tag{22}$$

From (21) and (22), using the definition (19), we obtain

$$[a_0, \ldots, a_{2k}] < \frac{\xi_0}{\xi_1} < [a_0, \ldots, a_{2k+1}]. \tag{23}$$

To show that the sequence of $[a_0, \ldots, a_k]$ converges to ξ_0/ξ_1, we define

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{24}$$

We take this to be meaningful, even when $k = -1$. We can even let k be as low as -2 by defining $a_{-1} = 0$ and replacing (24) with

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} a_{-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have some values thus:

k	-2	-1	0	1	2
p_k	0	1	a_0	a_0a_1+1	$a_0a_1a_2+a_0+a_2$
q_k	1	0	1	a_1	a_1a_2+1

By induction, p_k and q_k are prime to one another, since 1 and 0 are, and also

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} ab+c & b \end{pmatrix}.$$

Using this, when $k \geqslant 0$, we can write the last two factors in (24) as a single factor. This way, as from (18) we obtain (20) and then (22), so now we have

$$\frac{p_n}{q_n} = [a_0, \dots, a_n]. \tag{25}$$

Moreover, (24) gives us

$$\begin{pmatrix} p_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k+1} \\ 1 \end{pmatrix}. \tag{26}$$

Combining this with (24), we obtain

$$\begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k+1} & 1 \\ 1 & 0 \end{pmatrix}. \tag{27}$$

Table 9: Numbers from $\log_2 3$

n	-2	-1	0	1	2	3	4	5
a_n			1	1	1	2	2	3
p_n	0	1	1	2	3	8	19	65
q_n	1	0	1	1	2	5	12	41

Plugging back into (26) yields

$$\begin{pmatrix} p_{k+2} \\ q_{k+2} \end{pmatrix} = \begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} \begin{pmatrix} a_{k+2} \\ 1 \end{pmatrix}. \tag{28}$$

This holds, even when k is -1 or -2. Thus, when $(\xi_0, \xi_1) = (\log 3, \log 2)$, we obtain Table 9. The values of p_n and q_n there occurred as exponents in Table 8. Meanwhile, (27) gives us

$$p_{n+1}q_n - p_n q_{n+1} = \det \begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = (-1)^n. \tag{29}$$

Since in addition now

$$q_0 = 1, \qquad q_1 = a_1, \qquad q_{n+2} = a_{n+2}q_{n+1} + q_n,$$

so that

$$1 = q_0 < q_1 < q_2 < \cdots, \tag{30}$$

(29) gives us

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n+1}q_n}, \tag{31}$$

and so the sequence of p_k/q_k converges. It converges then to ξ_0/ξ_1 by (23).

The fractions $[a_0, \dots, a_k]$, namely p_k/q_k, are the **convergents** of ξ_0/ξ_1. When $0 \leqslant j < a_k - 1$, we may call $[a_0, \dots, a_{k-1}, j+1]$ a **semi-convergent,** as do Carey and Clampitt and others, or an **intermediate fraction,** as does Khinchin [23, page 14]. By the original definition (19) of continued fractions, as well as (23),

$$\left.\begin{gathered}[a_0, \dots, a_{2k-1}, 1] < \dots < [a_0, \dots, a_{2k}] < \frac{\xi_0}{\xi_1}, \\ [a_0, \dots, a_{2k}] < [a_0, \dots, a_{2k+1}, 1],\end{gathered}\right\} \tag{32}$$

since also, generally,

$$[a_0, \dots, a_{k-1}, a_k + 1] = [a_0, \dots, a_k, 1].$$

Likewise,

$$\begin{gathered}[a_0, \dots, a_{2k}, 1] > \dots > [a_0, \dots, a_{2k+1}] > \frac{\xi_0}{\xi_1}, \\ [a_0, \dots, a_{2k+1}] > [a_0, \dots, a_{2k+2}, 1].\end{gathered}$$

Table 10: Fractions for $\log_2 3$

$[1]$	$= 1/\ 1$
$[1, 1]$	$= 2/\ 1$
$[1, 1, 1]$	$= 3/\ 2$
$[1, 1, 1, 1]$	$= 5/\ 3$
$[1, 1, 1, 2]$	$= 8/\ 5$
$[1, 1, 1, 2, 1]$	$= 11/\ 7$
$[1, 1, 1, 2, 2]$	$= 19/12$
$[1, 1, 1, 2, 2, 1]$	$= 27/17$
$[1, 1, 1, 2, 2, 2]$	$= 46/29$
$[1, 1, 1, 2, 2, 3]$	$= 65/41$
$[1, 1, 1, 2, 2, 3, 1]$	$= 84/53$

Using (28), we can generalize (25) to include the intermediate fractions, thus:

$$\frac{(j+1)p_{k-1} + p_{k-2}}{(j+1)q_{k-1} + q_{k-2}} = [a_0, \ldots, a_{k-1}, j+1]. \tag{33}$$

This becomes (25) when $j + 1 = a_k$. For example, we can augment Table 9, obtaining Table 10, where horizontal lines mark the shifts from one side of $\log_2 3$ to the other. Generally, by repeated application of (18), then (27),

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} \xi_k \\ \xi_{k+1} \end{pmatrix}. \tag{34}$$

We have therefore

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} p_{k-1} & jp_{k-1} + p_{k-2} \\ q_{k-1} & jq_{k-1} + q_{k-2} \end{pmatrix} \begin{pmatrix} \xi_k - j\xi_{k+1} \\ \xi_{k+1} \end{pmatrix}, \tag{35}$$

which, when $j = a_k$, becomes (34) with k raised to $k+1$ and matrix entries interchanged.

Now we have the numbers obtained by the following procedure.

Step 1. Start with the interval $[0, \xi_0]$.

Step 2. Divide it into two unequal intervals at ξ_1.

Step $n+1$. From each of the larger intervals left in Step n, cut an interval equal to each of the smaller ones.

Comparing (33) and (35), remembering (16), we see that the sequence of numbers of intervals into which $[0, \xi_0]$ is divided at each step is precisely the sequence of numerators of convergents and intermediate fractions of ξ_0/ξ_1.

Likewise, by starting with $[0, \xi_1]$ and dividing it at ξ_2, we get the sequence of *denominators* of convergents and intermediate fractions, starting with $[a_0, 1]$.

That sequence of denominators will turn out to meet two additional conditions, concerning points that determine the associated division or partition of $[0, \xi_1]$.

First, the division is made by residues *modulo* ξ_1 of consecutive multiples of ξ_0. To show this more simply, we scale down by ξ_1, in the sense of replacing (ξ_0, ξ_1) with $(\alpha, 1)$. Fixing some positive integer n, when $1 \leqslant j < n$, and only then, we let the fractional part of $j\alpha$ be α_j. Thus,

$$\alpha_j = \{j\alpha\} = j\alpha - \lfloor j\alpha \rfloor. \tag{36}$$

Note that, when x is not an integer,

$$\{-x\} = 1 - \{x\}. \tag{37}$$

In some order, the points $\alpha_1, \ldots, \alpha_{n-1}$ partition $[0, 1]$ into n intervals. If $[\alpha_i, \alpha_j]$ is one of those intervals, its length is $\{j\alpha\} - \{i\alpha\}$. This is $\{(j - i)\alpha\}$, which is

- α_{j-i}, if $i < j$;
- $1 - \alpha_{i-j}$, if $i > j$, by (37).

Of the intervals of the partition, let the one

- that starts at 0 end at α_c;
- start at α_d that ends at 1.

Then among the n intervals of the partition are

- $n - c$ intervals having length α_c, namely

$$[0, \alpha_c], \quad [\alpha_1, \alpha_{c+1}], \quad \ldots, \quad [\alpha_{n-1-c}, \alpha_{n-1}];$$

- $n - d$ intervals having length $1 - \alpha_d$, namely

$$[\alpha_d, 1], \quad [\alpha_{d+1}, \alpha_1], \quad \ldots, \quad [\alpha_{n-1}, \alpha_{n-1-d}].$$

We have thus accounted for $2n - (c + d)$ of the n intervals. In particular,

$$n \leqslant c + d.$$

If any intervals remain, their number is $c+d-n$, and themselves are

$$[\alpha_{n-c}, \alpha_{n-d}], \qquad \ldots, \qquad [\alpha_{d-1}, \alpha_{c-1}],$$

and their length is $\alpha_c + 1 - \alpha_d$, since they can be partitioned respectively, at

$$\{n\alpha\}, \qquad \ldots, \qquad \{(c + d - 1)\alpha\},$$

into intervals of lengths α_c and $1 - \alpha_d$ respectively.

Thus, as was apparently first worked out in the 1950s [37], the residues of consecutive multiples of α, starting with α itself, partition $[0, 1]$ into intervals of just two or three lengths; moreover, the third length, if there is one, is the sum of the other two. This gives us what we want.

Having in mind the examples of the previous section, we might consider the case of two lengths as a "generalized scale," the number of its "notes" being the number of intervals.

For the remaining condition, due to Carey and Clampitt, on the points determining the division into those intervals, we look at how closely the convergents and intermediate fractions approximate ξ_0/ξ_1, which is α. Some of the following can be found for example in [22, page 70]. If

$$\frac{b}{d} < \frac{x}{y} < \frac{a}{c}, \tag{38}$$

all letters representing positive integers, then subtracting the first fraction yields

$$0 < \frac{1}{dy} \leqslant \frac{dx - by}{dy} < \frac{ad - bc}{cd}.$$

Suppose we know also $ad - bc = 1$, that is,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1. \tag{39}$$

Then $y > c$. Likewise, $y > d$, since

$$0 < \frac{bc - ad}{cd} < \frac{cx - ay}{cy} \leqslant \frac{1}{cy} < 0.$$

We still have (39) when we replace (a, c) or (b, d) with $(a+b, c+d)$. Here $(a+b)/(c+d)$ is the **mediant** of a/c and b/d, and it has the least denominator of all fractions between those two.

The foregoing applies to convergents and intermediate fractions of ξ_0/ξ_1, thanks to (29). By (25) and (33),

fractions		mediant
$[a_0, \dots, a_k]$	$[a_0, \dots, a_k, j+1]$	$[a_0, \dots, a_k, j+2]$
$[a_0, \dots, a_k]$	$[a_0, \dots, a_{k+1}]$	$[a_0, \dots, a_{k+1}, 1]$

Suppose (38) and (39) hold when a, b, c, and d are still positive integers, but x/y is ξ_0/ξ_1. By (32), (33), and convergence, for some even k and some j where $0 \leqslant j < a_k$,

$$\frac{jp_{k-1} + p_{k-2}}{jq_{k-1} + q_{k-2}} < \frac{b}{d} \leqslant \frac{(j+1)p_{k-1} + p_{k-2}}{(j+1)q_{k-1} + q_{k-2}} < \frac{\xi_0}{\xi_1} < \frac{a}{c}.$$

This shows

$$d = (j+1)q_{k-1} + q_{k-2}$$

and then b/d is the convergent or intermediate fraction of which d is the denominator. Likewise, a/c must be a convergent or intermediate fraction. More than that, since, in our original assumptions, we can replace b/d or a/c with the mediant of the two of them, this mediant too must be a convergent or intermediate fraction.

Again, for any positive integer n, the points $\{j\alpha\}$, where $1 \leqslant j < n$, in *some* order, partition $[0,1]$ into n intervals. For any integer t that is prime to n, the points $\{j\alpha\}$, collectively, are also $\{\overline{jt}\alpha\}$, where $\overline{x}$ is the residue of x in $\{0,\dots,n-1\}$ with respect to the modulus n. We shall identify those n for which there is some t in the set $\{1,\dots,n-1\}$ that ensures

$$\{\overline{t}\alpha\} < \{\overline{2t}\alpha\} < \dots < \{\overline{(n-1)t}\alpha\}. \tag{40}$$

In any case,

$$\sum_{j=0}^{n-2}\left(\{\overline{(j+1)t}\alpha\} - \{\overline{jt}\alpha\}\right) + \left(1 - \{\overline{(n-1)t}\alpha\}\right) = 1,$$

but (40) means all n terms in the sum here are positive. We may assume $n > 1$ and $\overline{t} = t$. We note then

$$\begin{aligned}\overline{(j+1)t} &= \begin{cases}\overline{jt} + t, & \text{if } \overline{jt} < n - t,\\ \overline{jt} + t - n, & \text{if } \overline{jt} \geqslant n - t,\end{cases}\\ \overline{(n-1)t} &= n - t.\end{aligned}$$

Now we can compute the following values, *if they are positive,* using (37) in the second case:

$$\{\overline{(j+1)t}\alpha\} - \{\overline{jt}\alpha\}$$
$$= \begin{cases} \{t\alpha\}, & \text{if } \overline{jt} < n-t, \\ 1 - \{(n-t)\alpha\}, & \text{if } \overline{jt} \geqslant n-t. \end{cases}$$

Also,

$$1 - \{\overline{(n-1)t}\alpha\} = 1 - \{(n-t)\alpha\}.$$

When $0 \leqslant j \leqslant n-2$, we have the condition

$$\overline{jt} < n - t$$

in exactly that many cases. Therefore (40) is equivalent to

$$(n-t)\{t\alpha\} + t\bigl(1 - \{(n-t)\alpha\}\bigr) = 1,$$

which we can rewrite as

$$\det\begin{pmatrix} 1 - \{(n-t)\alpha\} & -\{t\alpha\} \\ n-t & t \end{pmatrix} = 1. \tag{41}$$

Recalling (36), we see that (41) is equivalent to

$$\det\begin{pmatrix} \lfloor (n-t)\alpha \rfloor + 1 & \lfloor t\alpha \rfloor \\ n-t & t \end{pmatrix} = 1. \tag{42}$$

We note now always

$$\frac{\lfloor t\alpha \rfloor}{t} < \alpha < \frac{\lfloor (n-t)\alpha \rfloor + 1}{n-t}. \tag{43}$$

If (40) and therefore (42) hold, then as we have seen, the fractions in (43) and their mediant, whose denominator is n, must be denominators of convergents or semi-convergents.

Conversely, suppose n is the denominator of $[a_0, \dots, a_k, j+1]$, where $0 \leqslant j < a_{k+1}$. By (33),

$$n = (j+1)q_k + q_{k-1}. \tag{44}$$

Here,

- q_k is the denominator of $[a_0, \dots, a_k]$,
- $n - q_k$ is the denominator
  - of $[a_0, \dots, a_k, j]$, if $j > 0$;
  - of $[a_0, \dots, a_{k-1}]$, if $j = 0$, since in this case $k > 0$, because we assume $n > 1$.

Thus, in (43), when we let

$$t = \begin{cases} q_k, & \text{if } k \text{ is even,} \\ n - q_k, & \text{if } k \text{ is odd,} \end{cases}$$

then the fractions themselves must be the corresponding convergents and intermediate fractions. In this case, (42) and therefore (40) are true.

We have shown the following.

1. For all non-negative integers k, by (34),

$$\xi_1 = q_k \xi_{k+1} + q_{k-1} \xi_{k+2}$$

   and then, more generally, by (35), whenever $0 \leqslant j < a_{k+1}$,

$$\xi_1 = q_k(\xi_{k+1} - j\xi_{k+2}) + (jq_k + q_{k-1})\xi_{k+2}.$$

2. The pairs of lengths of intervals just indicated into which $[0, \xi_1]$ can be divided are just those achieved by *anthyphaeresis,* or alternate subtraction, starting with ξ_1 and ξ_2.
3. The least positive residues *modulo* ξ_1 of the first $n - 1$ positive multiples of ξ_0 divide $[0, \xi_1]$ into intervals of three lengths, unless (44) holds for some k and j, where $k \geqslant 0$ and $0 \leqslant j < a_{k+1}$, and then there are one or two lengths, as indicated.
4. Precisely when there are one or two lengths, the set of residues is **well-formed** in the sense of Carey and Clampitt: in the taking of the residues, the order of the multiples is changed according to an automorphism of the cyclic group of order n.

Thus the idea found in Plato and Euclid of making music with powers of two and three has been pursued into the twenty-first century.

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